

Change to "associative array" notation: $(x:\lambda x)$.
Use semi-colons to separate w.

Balloons and Hoops and their Universal Finite-Type Invariant, BF Theory, and an Ultimate Alexander Invariant

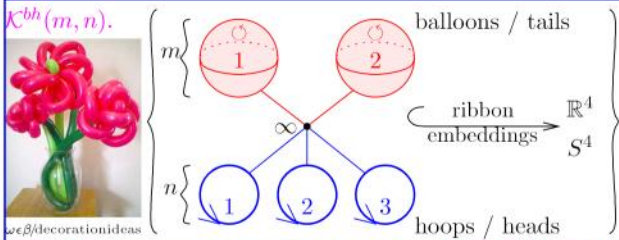
b" h

Dror Bar-Natan in Hamburg, August 2012
 ωεβ := http://www.math.toronto.edu/~drorbn/Talks/Hamburg-1208



Scheme. • Balloons and hoops in \mathbb{R}^4 , algebraic structure and relations with 3D.

- An ansatz for a “homomorphic” invariant: computable, related to finite-type and to BF.
- Reduction to an “ultimate Alexander invariant”.



Examples.

ϵ_x :

ϵ_u :

ρ_{ux}^+ :

ρ_{ux}^- :

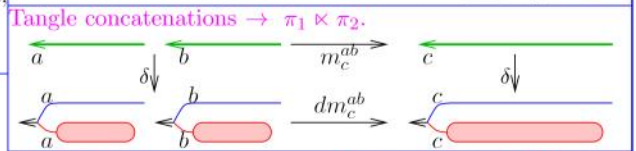
I mean business!

$T_0 = \text{Rn}(3, a) \text{Rp}(2, 2) \text{Rp}(1, 4)$
 $\mathcal{E} = T_0 // \text{dn}(2, 1, 1) // \text{dn}(4, b, b) //$
 $\text{dn}(1, a, a) // \text{dn}(3, a, a) //$
 $\mathcal{E}((9)) / (v_{\text{SW}} = \text{Deg}[w] + 1) w,$
 $w_{\text{CW}} = \text{Deg}[w] + w$

$w[\text{CW}] = [a], -2 [ab], -3 [aab], -3 [abb],$
 $-4 [aaab] - 42 [aabb] - 60 [abab] - 4 [abbb],$
 $-5 [aaaab] - 110 [aaabb] - 180 [aabbab] -$
 $110 [aabbba] - 180 [ababbb] - 5 [abbbb],$
 $b[2] \text{S}[2](a, 0, -24 [aaab],$
 $-60 [aaab] - 60 [aabb], -120 [aaab] -$
 $900 [aaab] - 360 [aabb] - 120 [aabb] -$
 $b[2] \text{S}[2](a, 2 [b], 9 [ab], 26 [aab] -$
 $24 [abb], 60 [aaab] - 255 [aabb] - 60 [abbb],$
 $119 [aaab] - 1504 [aaab] - 119 [aaab] -$
 $1504 [aabb] + 1384 [abbb] - 119 [abbbb])$

Meta-associativity.

$dm_a^{ab} // dm_b^{ac} = dm_b^{bc} // dm_a^{ab}$



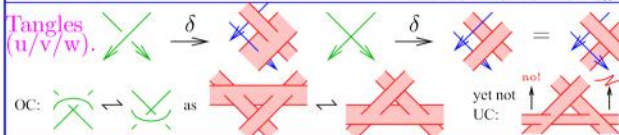
Thus we seek homomorphic invariants of \mathcal{K}^{bh} !

Invariant #0. With Π_1 denoting “honest π_1 ”, map $\gamma \in \mathcal{K}^{bh}(m, n)$ to the triple $(\Pi_1(\gamma^c), (u_i), (x_j))$, where the meridian of the balls u_i normally generate Π_1 , and the “longitudes” x_j are some elements of Π_1 .
 * acts like *, tm acts by “merging” two meridians/generators, hm acts by multiplying two longitudes, and hta^{xu} acts by “conjugating a meridian by a longitude”:

$(\Pi, (u, \dots), (x, \dots)) \mapsto (\Pi * \langle \bar{u} \rangle / (u = x \bar{u} x^{-1}), (\bar{u}, \dots), (x, \dots))$

Failure #0. Can we write the x 's as free words in the u 's?
 If $x = uv$, compute $x // hta^{xu}$:

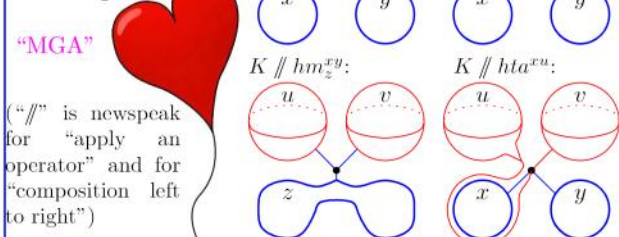
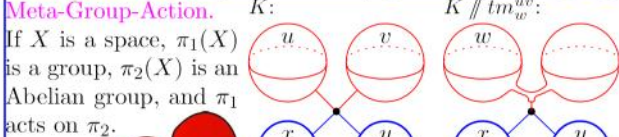
$x = uv \rightarrow \bar{u}v = u^x v = u^{\bar{u}v} v = u^{u^x v} v = u^{u^x v} v = \dots$



- δ injects u-Knots into \mathcal{K}^{bh} (likely u-tangles too).
- δ maps v/w-tangles map to \mathcal{K}^{bh} ; the kernel contains Reidemeister moves and the “overcrossings commute” relation, and conjecturally, that's all. Allowing punctures and cuts, δ is onto.



Operations
 Punctures & Cuts
 Connected Sums.



- Properties.**
- Associativities: $m_a^{ab} // m_a^{ac} = m_b^{bc} // m_a^{ab}$, for $m = tm, hm$.
 - Action axiom t : $tm_w^{uv} // hta^{xw} = hta^{xu} // hta^{xv} // tm_w^{uv}$,
 - Action axiom h : $hm_z^{xy} // hta^{zu} = hta^{xu} // hta^{yu} // hm_z^{xy}$.
 - SD Product: $dm_c^{ab} := hta^{ba} // tm_c^{ab} // hm_c^{ab}$ is associative.

The Meta-Group-Action M . Let T be a set of “tail labels” (“balloon colours”), and H a set of “head labels” (“hoop colours”). Let $FL = FL(T)$ and $FA = FA(T)$ be the (completed graded) free Lie and free associative algebras on generators T and let $CW = CW(T)$ be the (completed graded) vector space of cyclic words on T , so there's $\text{tr} : FA \rightarrow CW$. Let $M(T, H) := \{(\bar{\lambda} = (\lambda_x)_{x \in H}, \omega) : \lambda_x \in FL, \omega \in CW\}$

Operations. Set $(\bar{\lambda}_1; \omega_1) * (\bar{\lambda}_2; \omega_2) := (\bar{\lambda}_1 \bar{\lambda}_2; \omega_1 + \omega_2)$ and with $\mu = (\bar{\lambda}, \omega)$ define

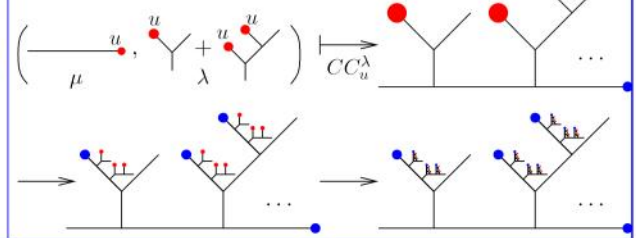
$tm_w^{uv} : \mu \mapsto \mu // (u, v \mapsto w),$

$hm_z^{xy} : \mu \mapsto ((\dots \bar{\lambda}_x \bar{\lambda}_y \dots) \text{bch}(\lambda_x, \lambda_y), \omega)$

“stable apply”

$hta^{xu} : \mu \mapsto \underbrace{\mu // (u \mapsto e^{\text{ad } \lambda_x}(\bar{u}))}_{\mu // CC_u^\lambda} // (\bar{u} \mapsto u) + (0, J_u(\lambda_x))$
 the “ J -spice”

A CC_u^λ example.



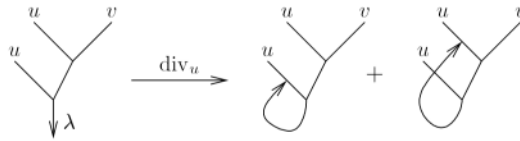
Balloons and Hoops and their Universal Finite-Type Invariant, 2

The Meta-Cocycle J . Set $J_u(\lambda) := J(1)$ where

$$J(0) = 0, \quad \lambda_s = \lambda \parallel CC_u^{s\lambda},$$

$$\frac{dJ(s)}{ds} = (J(s) \parallel \text{der}(u \mapsto [\lambda_s, u])) + \text{div}_u \lambda_s,$$

and where $\text{div}_u \lambda := \text{tr}(u\sigma_u(\lambda))$, $\sigma_u(v) := \delta_{uv}$, $\sigma_u([\lambda_1, \lambda_2]) := \iota(\lambda_1)\sigma_u(\lambda_2) - \iota(\lambda_2)\sigma_u(\lambda_1)$ and ι is the inclusion $FL \hookrightarrow FA$:



Claim. $CC_u^{\text{bch}(\lambda_1, \lambda_2)} = CC_u^{\lambda_1} \parallel CC_u^{\lambda_2} \parallel CC_u^{\lambda_1}$ and

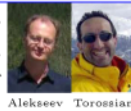
$$J_u(\text{bch}(\lambda_1, \lambda_2)) = J_u(\lambda_1) \parallel CC_u^{\lambda_2} \parallel CC_u^{\lambda_1} + J_u(\lambda_2 \parallel CC_u^{\lambda_1}),$$

and hence tm , hm , and hta form a meta-group-action.

Why ODEs? Q. Find f s.t. $f(x+y) = f(x)f(y)$.

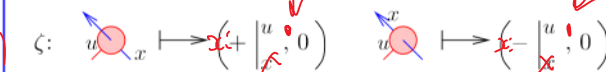
A. $\frac{df(s)}{ds} = \frac{d}{ds} f(s + \epsilon) = \frac{d}{ds} f(s)f(\epsilon) = f(s)C$.

Now solve this ODE using Picard's theorem or power series.



Alekseev Torossian

The Invariant ζ . Set $\zeta(\rho^\pm) = (\pm u_x, 0)$. This at least defines an invariant of $u/v/w$ -tangles, and if the topologists will deliver a "Reidemeister" theorem, it is well defined on \mathcal{K}^{bh} .



Theorem. ζ is (the log of) a universal finite type invariant (a homomorphic expansion) of w -tangles.

Tensorial Interpretation. Let \mathfrak{g} be a finite dimensional Lie algebra (any!). Then there's $\tau : FL(T) \rightarrow \text{Fun}(\oplus_T \mathfrak{g} \rightarrow \mathfrak{g})$ and $\tau : CW(T) \rightarrow \text{Fun}(\oplus_T \mathfrak{g})$. Together, $\tau : M(T, H) \rightarrow \text{Fun}(\oplus_T \mathfrak{g} \rightarrow \oplus_H \mathfrak{g})$, and hence

$$e^\tau : M(T, H) \rightarrow \text{Fun}(\oplus_T \mathfrak{g} \rightarrow \mathcal{U}^{\otimes H}(\mathfrak{g})).$$

ζ and BF Theory. Let A denote a \mathfrak{g} -connection on S^4 with curvature F_A , and B a \mathfrak{g}^* -valued 2-form on S^4 . For a hoop γ_x , let $\text{hol}_{\gamma_x}(A) \in \mathcal{U}(\mathfrak{g})$ be the holonomy of A along γ_x . For a ball γ_u , let $\mathcal{O}_{\gamma_u}(B) \in \mathfrak{g}^*$ be the integral of B (transported via A to ∞) on γ_u .



Cattaneo

Loose Conjecture. For $\gamma \in \mathcal{K}(T, H)$,

$$\int \mathcal{D}A \mathcal{D}B e^{\int B \wedge F_A} \prod_u e^{\mathcal{O}_{\gamma_u}(B)} \bigotimes_x \text{hol}_{\gamma_x}(A) = e^\tau(\zeta(\gamma)).$$

That is, ζ is a complete evaluation of the BF TQFT.

Issues. How exactly is B transported via A to ∞ ? How does the ribbon condition arise? Or if it doesn't, could it be that ζ can be generalized??

The β quotient, 1. • Arises when \mathfrak{g} is the 2D non-Abelian Lie algebra.

• Arises when reducing by relations satisfied by the weight system of the Alexander polynomial.



"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

www.katlas.org



The β quotient, 2. Let $R = \mathbb{Q}[\{c_u\}_{u \in T}]$ and $L_\beta := R \otimes T$ with central R and with $[u, v] = c_u v - c_v u$ for $u, v \in T$. Then $FL \rightarrow L_\beta$ and $CW \rightarrow R$. Under this,

$$\mu \rightarrow (\bar{\lambda}; \omega) \quad \text{with } \bar{\lambda} = \sum_{x \in H, u \in T} \lambda_{ux} u x, \quad \lambda_{ux}, \omega \in R,$$

$$\text{bch}(u, v) \rightarrow \frac{c_u + c_v}{e^{c_u + c_v} - 1} \left(\frac{e^{c_u} - 1}{c_u} u + e^{c_u} \frac{e^{c_v} - 1}{c_v} v \right),$$

if $\lambda = \sum \lambda_v v$ then with $c_\lambda := \sum \lambda_v c_v$,

$$u \parallel CC_u^\lambda = \left(1 + c_u \lambda_u \frac{e^{c_\lambda} - 1}{c_\lambda} \right)^{-1} \left(e^{c_\lambda} u - c_u \frac{e^{c_\lambda} - 1}{c_\lambda} \sum_{v \neq u} \lambda_v v \right),$$

$\text{div}_u \lambda = c_u \lambda_u$, and the ODE for J integrates to

$$J_u(\lambda) = \log \left(1 + \frac{e^{c_\lambda} - 1}{c_\lambda} c_u \lambda_u \right),$$

so ζ is formula-computable to all orders! Can we simplify?

Repackaging. Given $((\lambda_{ux}); \omega)$, set $c_x := \sum_v c_v \lambda_{vx}$, replace $\lambda_{ux} \rightarrow \alpha_{ux} := c_u \lambda_{ux} \frac{e^{c_x} - 1}{c_x}$ and $\omega \rightarrow \log \omega$, use $t_u = e^{c_u}$, and write α_{ux} as a matrix. Get " β calculus".

β Calculus. Let $\beta(H, T)$ be

$$\left\{ \begin{array}{c|ccc|c} \omega & x & y & \cdots & \omega \text{ and the } \alpha_{ux} \text{'s are} \\ u & \alpha_{ux} & \alpha_{uy} & \cdot & \text{rational functions in} \\ v & \alpha_{vx} & \alpha_{vy} & \cdot & \text{variables } t_u, \text{ one for} \\ \vdots & \cdot & \cdot & \cdot & \text{each } u \in T. \end{array} \right\},$$



In preparation, Seimani & B-N.

$$tm_{uv}^w : \frac{\omega}{u} \left| \begin{array}{c} \cdots \\ \alpha \\ \beta \\ \gamma \end{array} \right. \mapsto \frac{\omega}{w} \left| \begin{array}{c} \cdots \\ \alpha + \beta \\ \gamma \end{array} \right., \quad = \frac{\omega_1}{T_1} \left| \begin{array}{c} H_1 \\ \alpha_1 \\ 0 \end{array} \right. \cup \frac{\omega_2}{T_2} \left| \begin{array}{c} H_2 \\ \alpha_2 \\ 0 \end{array} \right.,$$

$$hm_z^{xy} : \frac{\omega}{\cdot} \left| \begin{array}{ccc} x & y & \cdots \\ \alpha & \beta & \gamma \end{array} \right. \mapsto \frac{\omega}{\cdot} \left| \begin{array}{ccc} z & \cdots \\ \alpha + \beta + \langle \alpha \rangle \beta & \gamma \end{array} \right.>,$$

$$hta^{xu} : \frac{\omega}{u} \left| \begin{array}{ccc} x & \cdots & \omega \epsilon \\ \alpha & \beta & \vdots \\ \gamma & \delta & \vdots \end{array} \right. \mapsto \frac{\omega \epsilon}{u} \left| \begin{array}{ccc} x & \cdots & \\ \alpha(1 + \langle \gamma \rangle / \epsilon) & \beta(1 + \langle \gamma \rangle / \epsilon) & \\ \gamma / \epsilon & \delta - \gamma \beta / \epsilon & \end{array} \right. ,$$

where $\epsilon := 1 + \alpha$, $\langle \alpha \rangle := \sum_v \alpha_v$, and $\langle \gamma \rangle := \sum_{v \neq u} \gamma_v$, and let

$$R_{ux}^+ := \frac{1}{u} \left| \begin{array}{c} x \\ t_u - 1 \end{array} \right. \quad R_{ux}^- := \frac{1}{u} \left| \begin{array}{c} x \\ t_u^{-1} - 1 \end{array} \right. .$$

On long knots, ω is the Alexander polynomial!

Why bother? (1) An ultimate Alexander invariant: Manifestly polynomial (time and size) extension of the (multi-variable) Alexander polynomial to tangles. Every step of the computation is the computation of the invariant of some topological thing (no fishy Gaussian elimination!). If should be an Alexander invariant to have an algebraic categorification, it is this one! See also $\omega \epsilon \beta / \text{regina}$, $\omega \epsilon \beta / \text{gwu}$.

Why bother? (2) Related to A-T, K-V, and E-K, should have vast generalization beyond w -knots and the Alexander polynomial. See also $\omega \epsilon \beta / \text{wko}$, $\omega \epsilon \beta / \text{caen}$, $\omega \epsilon \beta / \text{swiss}$.