Meta-Groups, Meta-Bicrossed Products and the Alexander Polynomial

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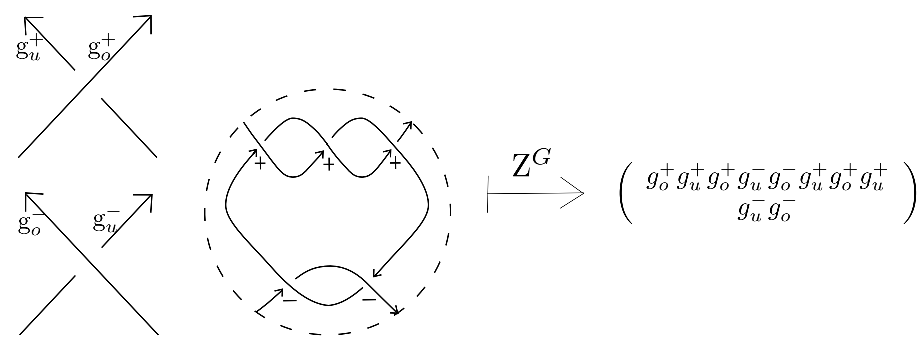
Abstract

Later

These are lecture notes for talks given by the first author, written and completed by the second. The talks, with handouts and videos, are available at [video]. See also further comments at [video].

1 Warm-up: the baby invariant, Z^G

Let T be an oriented tangle diagram. Let G be a group, and suppose we are given two pairs $R^{\pm} = (g_o^{\pm}, g_u^{\pm})$ of elements of G. At each positive (resp. negative)[1] crossing of T, assign g_o^+ (resp. g_o^-) to the upper strand and g_u^+ (resp. g_u^-) to the lower strand, as in Figure 1. Then, for every strand, multiply all elements assigned to it in the order that they appear and store the end result. If T has n strands, we get a collection of n elements of G. Call this collection $Z^G(T)$.



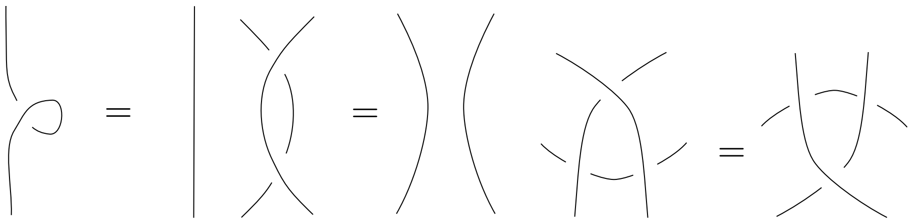
(a) assigning values to crossings

(b) collecting along strands

Figure 1: Computing Z^G of a tangle

Unfortunately, the gods are not so kind and Z^G is not worth much more than the effort that went in it. Indeed, invariance under the Reidemeister *II* move (see Figure 2) demands $g_o^- = (g_o^+)^{-1}$ and $g_u^- = (g_u^+)^{-1}$, while Reidemeister *III* adds that g_o^+ and g_u^+, as well as g_o^- and g_u^-, commute. As a result, every component of $Z^G(T)$ collapses to the form $g_o^a g_u^b$ for some integers a and b, so all the information to bring home is the

[1] Signs are determined by the "right-hand rule": If the right-hand thumb points along the direction of the upper strand of a positive crossing, then the fingers curl in the direction of the lower strand.



(a) Reidemeister I (b) Reidemeister II (c) Reidemeister III

Figure 2: The three Reidemeister moves

signed number of times a given strand crosses over or under ~~any~~ all other strand~~.~~s It will turn out, nevertheless, that a ~~similar~~ procedure yields an amply non-trivial invariant with novel properties.
a generalized version of this

2 A better invariant: Z^β

The invariant that we wish to introduce can be thought of as taking values in a meta-group. This is a generalization of what we call a "group computer":

2.1 Preliminary: A Group Computer

If X is a finite set and G is a group we let G^X denote the set of all possible assignments of elements of G to the set X; these are "G-valued datasets, with registers labelled by the elements of X".

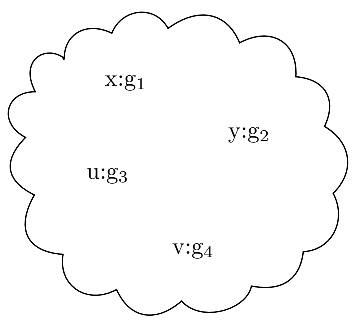


Figure 3: A typical element of $G^{\{x,y,u,v\}}$

if X does not contain the labels x, y, and z,

A group computer can manipulate registers in some prescribed ways. For example, define $m^z_{xy} : G^{X\cup\{x,y\}} \to G^{X\cup\{z\}}$ using the group multiplication, $\{x : g_1, y : g_2\} \mapsto \{z : g_1 g_2\}$. There are obvious ~~many~~ operations for inverting or cloning an element, renaming or deleting a register, and inserting the identity in a new register, respectively denoted S^x, Δ^x_{yz}, ρ^y_x, d_x and e^y, and respectively implemented on $G^{X\cup\{x\}}$ by fixing the content of X and mapping $\{x : g\}$ to $\{x : g^{-1}\}$, $\{y : g, z : g\}$, $\{y : g\}$, $\{\}$ and $\{x : g, y : e\}$. In addition there is a binary operation for merging data sets, $\bigcup : G^X \times G^Y \to G^{X\cup Y}$, which takes two data sets P and Q and ~~makes~~ their disjoint union $P \cup Q$.
Forms

2.2 Meta-Groups

The operations on a group computer obey a certain set of basic set-theoretic axioms as well as axioms inherited from the group G. A meta-group is an abstract computer that satisfies some but not all of those axioms. We postpone the precise definition to Section 3; it is best to begin with a prototypical example, as follows. Let $G_X := M_{X\times X}(\mathbf{Z})$ denote (not in reference to any group G) the set of $|X|\times|X|$ matrices of integers with rows and columns labelled by X. The operation of "multiplication", on say, 3×3 matrices, $m^{xy}_z : G_{\{x,y,w\}} \to G_{\{z,w\}}$, is defined by simultaneously adding rows and columns labelled by x and y:

$$\begin{array}{c c} & \begin{array}{ccc} x & y & w \end{array} \\ \begin{array}{c} x \\ y \\ w \end{array} & \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \end{array} \mapsto \begin{array}{c c} & \begin{array}{cc} z & w \end{array} \\ \begin{array}{c} z \\ w \end{array} & \begin{pmatrix} a+b+d+e & c+f \\ g+h & i \end{pmatrix} \end{array}$$

use and explain the // notation

While still satisfying the associativity condition $m^{xy}_u m^{uv}_w = m^{yv}_u m^{xu}_w$, this example differs from a group computer by the failure of a critical axiom: if $P \in G_{\{x,y\}}$,

$$d_yP \cup d_xP \neq P$$

Indeed, if $P \in G_{\{x,y\}}$ is the matrix $\begin{array}{cc} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} x \\ y \end{array} & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array}$, then

$$d_yP \cup d_xP = \begin{array}{cc} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} x \\ y \end{array} & \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \end{array} \neq P$$

There should be an informal definition of a meta-group here: $\{G_X\}$, $m^z_{xy}: G_{X\cup\{x,y\}} \to ..$, satisfying associativity $m^{xy}_u m^{uz}_w = ..$, and other axioms to be specified later.

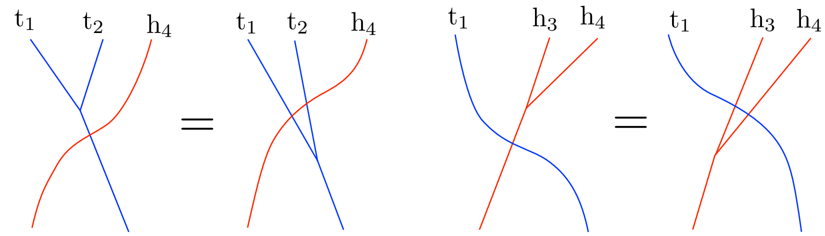
2.3 Meta-Bicrossed Products

Suppose a group G is given as the product $G = TH$ of two of its subgroups, where $T \cap H = \{e\}$. Then also $G = HT$ [2] and every element of G has unique[3] representations of the form th and $h't'$ where $h, h' \in H$ and $t, t' \in T$. Accordingly there is a "swap" map $sw : T \times H \to H \times T$, $(t,h) \mapsto (h',t')$ such that if $g = th$ then $g = h't'$ also. The swap map satisfies some relations; in group-computer language, the important ones are as in Figure 4 [4]. Conversely, provided that the swap map satisfies relations 4a and 4b, the data (H, T, sw) determines a group G, with product given by $\{(h_1,t_1),(h_2,t_2)\} \mapsto (h_1h'_2, t'_1t_2)$ where $sw(t_1,h_2) = (h'_2,t'_1)$. G is called the bicrossed product of H and T, which we will denote $(H\times T)_{sw}$. In a semidirect product, one of H or T is normal (say T) and the swap map is $sw : (t,h) \mapsto (h, h^{-1}th)$.

[2] Indeed, if $g^{-1} = th$, then $g = h^{-1}t^{-1}$, so $g^{-1} \in TH$ implies $g \in HT$, and as $TH = G$, also $HT = G$.

[3] Separation of variables: suppose $g = h_1t_1 = h_2t_2$. Then we have $h_2^{-1}h_1 = t_2t_1^{-1}$, which implies that $h_1 = h_2$ and $t_1 = t_2$ since $h_2^{-1}h_1 \in H$, $t_2t_1^{-1} \in T$, and $H \cap T = \{e\}$.

[4] $X \mathbin{/\!\!/} A \mathbin{/\!\!/} B$ is composition made unambiguous: first apply A to X, then apply B.



(a) $tm_1^{12} /\!\!/ sw_{14} = sw_{24} /\!\!/ sw_{14} /\!\!/ tm_1^{12}$ (b) $hm_3^{34} /\!\!/ sw_{13} = sw_{13} /\!\!/ sw_{14} /\!\!/ hm_3^{34}$

Figure 4: swap operation axioms. tm and hm stand for multiplication in T and H respectively.

M, T so as not to confuse with the groups from the previous page.

X X

The corresponding notion of ~~a~~ meta-bicrossed product is a collection of sets $\beta(\cancel{H},\cancel{T})$ indexed by all *pairs* of finite sets H and T (H for "heads", T for "tails"), and equipped with multiplication maps tm_z^{xy} (x, y and z tail labels), hm_z^{xy} (x, y and z head labels), and a swap map sw_{xy}^{th} (where t and h indicate that x is a tail label and y is a head label — note that sw_{yx}^{ht} is in general a different map) satisfying (a) and (b).
Given the above we can make a "group multiplication" map out of the head and tail multiplication maps via $gm_z^{xy} := sw_{xy}^{th} /\!\!/ tm_z^{xy} /\!\!/ hm_z^{xy}$. Thus a meta-bicrossed product defines a meta-group with $\Gamma_X = \beta(X, X)$. A prototypical example is again given by (now rectangular) matrices, $\mu(H,T) := M_{T\times H}(\mathbf{Z})$, with tm_z^{xy} and hm_z^{xy} corresponding to adding two rows and adding two columns, and swap being the trivial operation.

$$\begin{tikzcd}
 & \begin{array}{cc} & h_1 \\ \begin{matrix} t_1 \\ t_2 \\ t_3 \end{matrix} & \begin{pmatrix} a+b \\ c+d \\ e+f \end{pmatrix} \end{array} \\
\begin{array}{cc} & \begin{matrix} h_1 & h_2 \end{matrix} \\ \begin{matrix} t_1 \\ t_2 \\ t_3 \end{matrix} & \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \end{array} \arrow[ur, "hm_1^{1,2}"] \arrow[r, "tm_1^{1,2}"] \arrow[dr, "sw_{1,2}^{th}"'] & \begin{array}{cc} & \begin{matrix} h_1 & h_2 \end{matrix} \\ \begin{matrix} t_1 \\ t_3 \end{matrix} & \begin{pmatrix} a+c & b+d \\ e & f \end{pmatrix} \end{array} \\
 & \begin{array}{cc} & \begin{matrix} h_1 & h_2 \end{matrix} \\ \begin{matrix} t_1 \\ t_2 \\ t_3 \end{matrix} & \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \end{array}
\end{tikzcd}$$

lower case or upper case? And where is it?

2.4 β Calculus

The β calculus has an arcane origin [VIDEO][5] which is not appropriate to mention here. We expect that it can be presented in a much simpler ~~and fitting~~ context than that in which it was discovered. Accordingly we will simply pull it out of a hat. Let $\beta(H,T)$ be (again, in reference to *sets* H and T) the collection of arrays with rows labeled by $t_i \in T$ and columns labeled by $h_j \in H$, along with a distinguished element ω. Such arrays are conveniently presented in the following format:

$$\begin{array}{c|ccc} \omega & h_1 & h_2 & \dots \\ \hline t_1 & \alpha_{11} & \alpha_{12} & \cdot \\ t_2 & \alpha_{21} & \alpha_{22} & \cdot \\ \vdots & \cdot & \cdot & \cdot \end{array}$$

The α_{ij} and ω are rational functions of variables T_i, which are in bijection with the row labels t_i.

[5]in which, among other things, the "heads and tails" vocabulary is motivated.

$\beta(H,T)$ is equipped with a peculiar set of operations. Despite being repulsive at sight, they are completely elementary. They are defined as follows:

$$tm^{xy}_z : \begin{array}{c|c} \omega & \dots \\ \hline t_x & \alpha \\ t_y & \beta \\ \vdots & \gamma \end{array} \mapsto \begin{array}{c|c} \omega & \dots \\ \hline t_z & \alpha+\beta \\ \vdots & \gamma \end{array}$$

Here α and β are rows and γ is a matrix. The sum $\alpha+\beta$ is accompanied by the necessary change of variables T_x, $T_y \mapsto T_z$.

$$hm^{xy}_z : \begin{array}{c|ccc} \omega & h_x & h_y & \dots \\ \hline \vdots & \alpha & \beta & \gamma \end{array} \mapsto \begin{array}{c|cc} \omega & h_z & \dots \\ \hline \vdots & \alpha+\beta+\langle\alpha\rangle\beta & \gamma \end{array}$$

Here α and β are columns, γ is a matrix, and $\langle\alpha\rangle = \sum_i \alpha_i$.

$$sw^{th}_{xy} : \begin{array}{c|cc} \omega & h_y & \dots \\ \hline t_x & \alpha & \beta \\ \vdots & \gamma & \delta \end{array} \mapsto \begin{array}{c|cc} \omega\epsilon & h_y & \dots \\ \hline t_x & \alpha(1+\langle\gamma\rangle/\epsilon) & \beta(1+\langle\gamma\rangle/\epsilon) \\ \vdots & \gamma/\epsilon & \delta-\gamma\beta/\epsilon \end{array}$$

Here α is a single entry, β is a row, γ is a column, and δ is a matrix comprised of the rest. $\epsilon = 1+\alpha$. Note also that $\gamma\beta$ is the matrix product of the column γ with the row β and hence has the same dimensions as the matrix δ.

We also need the disjoint union, defined by

$$\begin{array}{c|c} \omega_1 & H_1 \\ \hline T_1 & \alpha_1 \end{array} \cup \begin{array}{c|c} \omega_1 & H_1 \\ \hline T_1 & \alpha_1 \end{array} = \begin{array}{c|cc} \omega_1\omega_2 & H_1 & H_2 \\ \hline T_1 & \alpha_1 & 0 \\ T_2 & 0 & \alpha_2 \end{array}$$

Finally there are two elements which will serve as a pair of "R-matrices", analogous to the pair of pairs $(g_o^\pm, g_u^\pm)$ of Z^G:

$$R^+_{xy} = \begin{array}{c|cc} 1 & h_x & h_y \\ \hline t_x & 0 & T_x - 1 \\ t_y & 0 & 0 \end{array} \qquad R^-_{xy} = \begin{array}{c|cc} 1 & h_x & h_y \\ \hline t_x & 0 & T_x^{-1} - 1 \\ t_y & 0 & 0 \end{array}$$

We make β into a meta-group via the "group-multiplication" map $gm^{xy}_z := sw_{xy} \mathbin{/\mkern-6mu/} tm^{xy}_z \mathbin{/\mkern-6mu/} hm^{xy}_z$. We will later set out to make proper definitions, write down the remaining operations, and establish the following

Theorem 1. *β is a meta-bicrossed product.*

2.5 Z^β

Let T be again an oriented tangle diagram. At each crossing, assign a number to the upper strand and to the lower strand. Form the disjoint union $\bigcup_{\{i,j\}} R^\pm_{ij}$ where $\{i,j\}$ runs over all pairs assigned to crossings, with i labelling the upper strand and j labelling the lower strand, and where $\pm$ is determined by the sign of the given crossing. Now for each strand multiply all the labels in the order in which they appear. That is, if the first label on the strand is k, apply gm^{kl}_k where l runs over all labels subsequently encountered on the strand (in order). If T has n strands, the result is an $n \times n$ array with corner element. Call this array $Z^\beta(T)$.

?

As an example, for the knot 8_{17} [ROLFSEN] illustrated in Figure 5a, make the disjoint union[6]

$$R^-_{12,1}R^-_{2,7}R^-_{8,3}R^-_{4,11}R^+_{16,5}R^+_{6,13}R^+_{14,3}R^+_{10,15},$$

which is given by the following array:

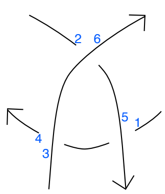
1	h_1	h_3	h_5	h_7	h_9	h_{11}	h_{13}	h_{15}
t_2	0	0	0	$T_2^{-1}-1$	0	0	0	0
t_4	0	0	0	0	0	$T_4^{-1}-1$	0	0
t_6	0	0	0	0	0	0	T_6-1	0
t_8	0	$T_8^{-1}-1$	0	0	0	0	0	0
t_{10}	0	0	0	0	0	0	0	$T_{10}-1$
t_{12}	$T_{12}^{-1}-1$	0	0	0	0	0	0	0
t_{14}	0	0	0	0	$T_{14}-1$	0	0	0
t_{16}	0	0	$T_{16}-1$	0	0	0	0	0

Then apply the multiplications gm_1^{1k}, with k running from 1 to 16, to get the following 1×1 array with corner element:

$-T_1^{-3}+4T_1^{-2}-8T_1^{-1}+11-8T_1+4T_1^2-T_1^3$	h_1
t_1	0

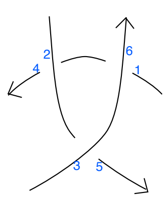
Theorem 2. *(1) Z^β is an invariant of oriented tangle diagrams. (2) Restricted to braids, it is equivalent to the Burau representation. (3) Restricted to knots, the corner element is the Alexander polynomial.*

Proof. (1) Trivial. We do the computation for the Reidemeister *III* move to illustrate. The disjoint unions for each side of the equality are given by:



$R^-_{1,5}R^-_{6,2}R^+_{3,4} =$

1	h_1	h_2	h_4
t_3	0	0	T_3-1
t_5	$T_5^{-1}-1$	0	0
t_6	0	$T_6^{-1}-1$	0



$R^+_{6,1}R^-_{2,4}R^-_{3,5} =$

1	h_1	h_4	h_5
t_3	0	$T_2^{-1}-1$	0
t_5	0	0	$T_3^{-1}-1$
t_6	T_6-1	0	0

Then one checks that indeed

$$R^-_{1,5}R^-_{6,2}R^+_{3,4}\,/\!\!/\, gm_1^{1,4}\,/\!\!/\, gm_2^{2,5}\,/\!\!/\, gm_3^{3,6} = R^+_{6,1}R^-_{2,4}R^-_{3,5}\,/\!\!/\, gm_1^{1,4}\,/\!\!/\, gm_2^{2,5}\,/\!\!/\, gm_3^{3,6} = \begin{array}{c|cc} 1 & h_1 & h_2 \\ \hline t_1 & T_2^{-1}-1 & 0 \\ t_2 & T_2^{-1}(T_3-1) & T_3^{-1}-1 \end{array}$$

(2) (SKETCH) The key is to show that a single crossing gets mapped to (essentially) its Burau representation via Z^β and that concatenating braids (essentially) corresponds to matrix multiplication of the "matrix part" of Z^β. ?

[6] From now on we omit the $\cup$ in disjoint unions: $\beta_1\beta_2 := \beta_1\cup\beta_2$. We also suppress rows/columns of zeros.

(3) (WISHFUL) Note first that a priori Z^β does not make sense on round knots, as one would have to multiply all the labels together (including the last remaining one with itself). It turns out however that round knots are equivalent to long knots, so one can simply pick an arbitrary[7] point to cut the knot open and compute Z^β without ambiguity. However to prove the assertion, given a knot K, turn it first into a braid b_K via Alexander's theorem. Compute $Z^\beta(b_K)$ to obtain the Burau representation, as in (2). Then multiply all the strands together except the last one; this operation is equivalent to taking the determinant of an $(n-1)\times(n-1)$ minor. □

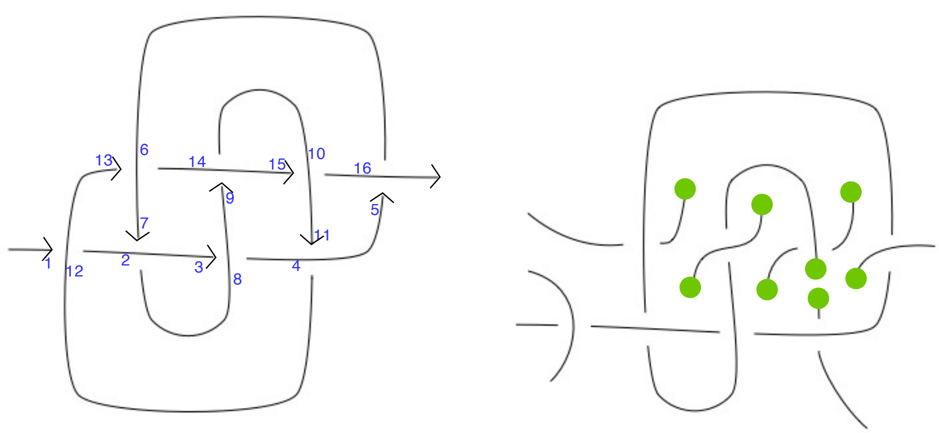
1. too much missing!
2. I'd much prefer to avoid braids.

One philosophically appealing major property of Z^β is that the operations used to compute it have a literal interpretation of gluing crossings together. In particular, at every stage of the computation we get an invariant of the tangle[8] made of all the crossings but only those for which the corresponding gm was carried out have been glued. Additionally, unlike other existing extensions of the Alexander polynomial to tangles, Z^β takes values in spaces of polynomial size, at every step of the calculation.

3 More on meta-groups

3.1 The meta-group of coloured v-tangles

When one tries to follow the interpretation of the computation of Z^β as progressively attaching crossings together to form a tangle, one will in general encounter a step where the tangle becomes non-planar (a strand will have to go through another in an "artificial" crossing to reach the boundary disk). See Figure 5b. Such tangles are called virtual or v-tangles and constitute a rich subject of study on their own [REF]. For us it will suffice to give them a name.



(a) 8_{17} with crossings labelled

(b) 8_{17} after attaching crossings 1 through 10. The arcs with green dots can not make it out to the boundary disk.

Figure 5: The knot 8_{17}

Armed with this new word in our vocabulary we can now define what seems to be the "most natural" meta-group: the meta-group of oriented coloured v-tangles. Let Γ_X be the set of v-tangles with strands labelled by X. There is a natural definition for all the meta-group operations. m_z^{xy} [9] concatenates strand x with

[7] ?

[8] the careful reader may wish to peek ahead at Section 3.1 for a better grasp of this statement.

strand y and labels the resulting strand z (note that we *need* virtual tangles for this to be well-defined), S^x reverses the orientation of strand x, e^x creates an isolated strand with label x, d^x deletes strand x, and Δ^x_{yz} is the cabling operation with input strand x and output strands y and z. [EXPAND]

also note that Δ^x_{yz} is not "commutative", and make sense only for framed tangles.

3.2 Definitions

We now proceed to laying down the details of the definitions of meta-groups and meta-bicrossed products. A more elegant approach is possible and partially realized in the next section.

A meta-group is a collection of sets Γ indexed by all finite sets, equipped with operations $m^{xy}_z : \Gamma_{\{x,y\}\cup X} \to \Gamma_{\{z\}\cup X}$, $S_x : \Gamma_X \to \Gamma_X$, $e_x : \Gamma_X \to \Gamma_{\{x\}\cup X}$, $d_x : \Gamma_{\{x\}\cup X} \to \Gamma_X$, $\Delta^x_{yz} : \Gamma_{\{x\}\cup X} \to \Gamma_{\{x,y\}\cup X}$, and $\bigcup : \Gamma_X \times \Gamma_Y \to \Gamma_{X\cup Y}$ satisfying the following:

- "Group theory axioms"
  - $e_x \,/\!\!/\, m^{xy}_z = \rho^y_z$ (left identity)
  - $m^{xy}_u \,/\!\!/\, m^{uz}_v = m^{yz}_u \,/\!\!/\, m^{xu}_v$ (associativity)
  - $\Delta^x_{y,z} \,/\!\!/\, S_y \,/\!\!/\, m^{yz}_x = d_x \,/\!\!/\, e_x$ (left inverse)
- "Set manipulation axioms"
  - $\rho^y_x \,/\!\!/\, \rho^x_y = id$
  - $\rho^x_y \,/\!\!/\, \rho^y_z = \rho^x_z$
  - $\Delta^x_{yz} \,/\!\!/\, d_y = \rho^x_z$
  - $\Delta^x_{yz} \,/\!\!/\, d_x = \rho^x_y$
  - $\rho^y_x \,/\!\!/\, d_y = d_x$
  - $m^{xy}_z \,/\!\!/\, d_z = d_x \,/\!\!/\, d_y$
  - $e_x \,/\!\!/\, d_x = id$
  - $S_x \,/\!\!/\, d_x = id$
  - $\Delta^x_{yz} = \Delta^x_{zy}$
  - $\Delta^x_{yz} \,/\!\!/\, \rho^z_u = \Delta^x_{yu}$
  - $\rho^x_u \,/\!\!/\, \Delta^u_{yz} = \Delta^x_{yz}$
  - $m^{xy}_z \,/\!\!/\, \rho^z_u = m^{xy}_u$
  - $\rho^x_u \,/\!\!/\, m^{uy}_z = m^{xy}_z$
  - $e_x \,/\!\!/\, \rho^x_y = e_y$
  - $S_x \,/\!\!/\, \rho^x_y = \rho^x_y \,/\!\!/\, S_y$
  - {operations involving disjoint sets of labels commute (e.g. $e_x \,/\!\!/\, e_y = e_y \,/\!\!/\, e_x$)}

two columns

A meta-bicrossed product is a collection of sets Γ indexed by all pairs of finite sets, equipped with maps hm, tm, and sw, such that:

- $hm^{h_x h_y}_{h_z} : \Gamma(H\cup\{h_x,h_y\},T) \to \Gamma(H\cup\{h_z\},T)$ and $tm^{t_x t_y}_{t_z} : \Gamma(H, T\cup\{t_x,t_y\}) \to \Gamma(H, T\cup\{t_z\})$ define meta-groups η and τ for each particular particular $t \in T$ and $h \in H$ respectively.

[9] Remark: this is *not* a meta-generalization of the group structure on braids.

- $m_z^{xy} = sw_{xy} \mathbin{/\!/} hm_{h_z}^{h_x h_y} \mathbin{/\!/} tm_{t_z}^{t_x t_y}$ defines a meta-group with $\Gamma_X = \Gamma(X, X)$
- sw_{xy} satisfies the following relations
  - $tm_x^{xy} \mathbin{/\!/} sw_{xz} = sw_{xz} \mathbin{/\!/} sw_{yz} \mathbin{/\!/} tm_x^{xy}$
  - $hm_y^{yz} \mathbin{/\!/} sw_{xy} = sw_{xy} \mathbin{/\!/} sw_{xz} \mathbin{/\!/} hm_y^{yz}$
  - $sw_{xy} \mathbin{/\!/} t\rho_u^x = t\rho_u^x \mathbin{/\!/} sw_{uy}$
  - $sw_{xy} \mathbin{/\!/} h\rho_u^y = h\rho_u^y \mathbin{/\!/} sw_{xu}$
  - $te_x \mathbin{/\!/} sw_{xy} = te_x$
  - $he_y \mathbin{/\!/} sw_{xy} = he_y$

If in addition all head operations commute with all tail operations, we call Γ a Hopf meta-bicrossed product (Hopf, because in the β calculus it is the doubling map that gives trouble, see Section 4.1).

3.3 An alternative approach

There is a curious way to package the definition of a meta-group in a much tidier fashion using categorical language, as follows. Let Op and $mset$ be the multicategories whose objects are finite sets and such that

$$Hom_{Op}(X^1, \dots, X^n, Y) = Hom_{Group}(FY \to F(X^1 \cup \dots \cup X^n))$$
$$Hom_{mset}(\Xi^1, \dots, \Xi^n \to \Theta) = Hom_{set}(\prod_{i=1}^{n} \Xi^i \to \Theta)$$

Here FX denotes the free group on the alphabet X. We can now give a very simple definition: a meta-group is a functor $\Gamma : Op \to mset$. The operations of Section 2 then correspond via Γ to free group homomorphisms given as follows:

$m_z^{xy} : F(\{z\} \cup X) \to F(\{x, y\} \cup X)$, $m_z^{xy}(a) = \begin{cases} xy & a = z \\ a & a \neq z \end{cases}$

$S^x : F(\{x\} \cup X) \to F(\{x\} \cup X)$, $S^x(a) = \begin{cases} a^{-1} & a = x \\ a & a \neq x \end{cases}$

$e_x : F(\{x\} \cup X) \to FX$, $e_x(a) = \begin{cases} e & a = x \\ a & a \neq x \end{cases}$

$\rho_y^x : F(\{y\} \cup X) \to F(\{x\} \cup X)$, $\rho_y^x(a) = \begin{cases} x & a = y \\ a & a \neq y \end{cases}$

$d_x : FX \to F(\{x\} \cup X)$, $d_x(a) = a$

$\Delta_{yz}^x : F(\{y, z\} \cup X) \to F(\{x\} \cup X)$, $\Delta_{yz}^x(a) = \begin{cases} x & a = y, a = z \\ a & \text{otherwise} \end{cases}$

$\bigcup_{X,Y} : F(X \cup Y) \to FX \times FY$

The axioms listed in the previous section then follow from identities satisfied by the above homomorphisms. For example, the axiom for the identity element insertion, $e_x \mathbin{/\!/} m_z^{xy} = \rho_z^y$ (the statement $eg = g$ for groups) follows from the calculation $e_x(m_z^{xy}(z)) = e_x(xy) = e_x(x)e_x(y) = ey = y = \rho_z^y(z)$.

It is slightly more challenging to find a "universal" model for meta-bicrossed products. One can ask whether there is a systematic way to turn any algebraic structure into a meta-algebraic structure. We will leave these matters open.

4 Some verifications: computer program

As mentioned and made explicit above, the operations of the β calculus are ugly. However, it can be implemented in a computer program in a very short paragraph, and the program handles the proofs of Theorems 1 and 2 very well. The following *Mathematica* code produces a ready-to-use program with neatly formatted output:

Initialization

```
βSimp = Factor; SetAttributes[βCollect, Listable];
βCollect[B[ω_, Λ_]] := B[βSimp[ω],
   Collect[Λ, h_, Collect[#, t_, βSimp] &]];
βForm[B[ω_, Λ_]] := Module[{ts, hs, M},
   ts = Union[Cases[B[ω, Λ], (t | T)_s_ :> s, Infinity]];
   hs = Union[Cases[B[ω, Λ], h_s_ :> s, Infinity]];
   M = Outer[βSimp[Coefficient[Λ, h_#1 t_#2]] &, hs, ts];
   PrependTo[M, t_# & /@ ts];
   M = Prepend[Transpose[M], Prepend[h_# & /@ hs, ω]];
   MatrixForm[M]];
βForm[else_] := else /. β_B :> βForm[β];
Format[β_B, StandardForm] := βForm[β];
B /: B[ω1_, β1_] == B[ω2_, β2_] := (ω1 == ω2) && (β1 == β2);
```

Program

```
⟨μ_⟩ := μ /. t_ → 1;
tm_{x_,y_→z_}[β_] := β /. {t_{x|y} → t_z, T_{x|y} → T_z};
hm_{x_,y_→z_}[B[ω_, Λ_]] := Module[
   {α = D[Λ, h_x], β = D[Λ, h_y], γ = Λ /. h_{x|y} → 0},
   B[ω, (α + (1 + ⟨α⟩) β) h_z + γ] // βCollect];
sw_{x_,y_}[B[ω_, Λ_]] := Module[{α, β, γ, δ, ϵ},
   α = Coefficient[Λ, h_y t_x]; β = D[Λ, t_x] /. h_y → 0;
   γ = D[Λ, h_y] /. t_x → 0;    δ = Λ /. h_y | t_x → 0;
   ϵ = 1 + α;
   B[ω * ϵ, α (1 + ⟨γ⟩ / ϵ) h_y t_x + β (1 + ⟨γ⟩ / ϵ) t_x
          + γ / ϵ h_y               + δ - γ * β / ϵ
    ] // βCollect];
gm_{x_,y_→z_}[β_] := β // sw_{x,y} // hm_{x,y→z} // tm_{x,y→z};
tΔ_{x_→y_,z_}[β_] := β /. {t_x → t_y + t_z, T_x → T_y T_z};
hΔ_{x_→y_,z_}[β_] := β /. {h_x → h_y + h_z};
B /: B[ω1_, Λ1_] B[ω2_, Λ2_] := B[ω1 * ω2, Λ1 + Λ2];
Rp_{x_,y_} := B[1, (T_x - 1) t_x h_y];
Rm_{x_,y_} := B[1, (T_x^-1 - 1) t_x h_y];
```

In the above, a β matrix is represented as a polynomial in two variables $\mu = \sum \alpha_{ij} t_i h_j$. This makes some calculations very simple! Selecting the content of column i is achieved by taking a derivative with respect to h_i; setting all the t's equal to 1 computes its column sum. The disjoint union of two matrices is simply the sum of their polynomials.

4.1 Theorem 1

To establish Theorem 1 we just need to check that the operations of β calculus satisfy the axioms of a meta-bicrossed product listed in Section 3.2. Let us illustrate the method with the important swap map axiom $tm_1^{12} \mathbin{/\!\!/} sw_{14} = sw_{14} \mathbin{/\!\!/} sw_{24} \mathbin{/\!\!/} tm_1^{12}$. We can use the computer program to check it on an arbitrary 3×2 array (i.e. an array with one more than the number of "participating" indices of each type):

```
{β = B[ω, Sum[α10 i+j ti hj, {i, {1, 2, 3}}, {j, {4, 5}}]],
 β1 = β // tm1,2→1 // sw1,4;
 β2 = β // sw2,4 // sw1,4 // tm1,2→1,
 FullSimplify[β1] == FullSimplify[β2]
}
```

$$\left\{\begin{pmatrix} \omega & h_4 & h_5 \\ t_1 & \alpha_{14} & \alpha_{15} \\ t_2 & \alpha_{24} & \alpha_{25} \\ t_3 & \alpha_{34} & \alpha_{35} \end{pmatrix}, \begin{pmatrix} \omega\,(1+\alpha_{14}+\alpha_{24}) & h_4 & h_5 \\ t_1 & \frac{(\alpha_{14}+\alpha_{24})\,(1+\alpha_{14}+\alpha_{24}+\alpha_{34})}{1+\alpha_{14}+\alpha_{24}} & \frac{(\alpha_{15}+\alpha_{25})\,(1+\alpha_{14}+\alpha_{24}+\alpha_{34})}{1+\alpha_{14}+\alpha_{24}} \\ t_3 & \frac{\alpha_{34}}{1+\alpha_{14}+\alpha_{24}} & \frac{-\alpha_{15}\,\alpha_{34}-\alpha_{25}\,\alpha_{34}+\alpha_{35}+\alpha_{14}\,\alpha_{35}+\alpha_{24}\,\alpha_{35}}{1+\alpha_{14}+\alpha_{24}} \end{pmatrix}, \text{True}\right\}$$

We claim that this constitutes a proof that the identity holds on arrays of arbitrary dimension. The key lies in the fact that the operations are linear in the "non-participating" indices. It is very clear then, from the 2-variable polynomial point of view, that the result still holds if one replaces a non-participating entry by an arbitrary sum. The argument applies to the other axioms as well and the reader is welcome to verify them.

As it stands, the β calculus is not a Hopf meta-bicrossed product. It is readily seen that doubling tails does not commute with multiplying heads:

```
{β = B[ω, a t1 h1 + b t1 h2],
 β // hm1,2→1 // tΔ1→1,2,
 β // tΔ1→1,2 // hm1,2→1
}
```

$$\left\{\begin{pmatrix} \omega & h_1 & h_2 \\ t_1 & a & b \end{pmatrix}, \begin{pmatrix} \omega & h_1 \\ t_1 & a+b+a\,b \\ t_2 & a+b+a\,b \end{pmatrix}, \begin{pmatrix} \omega & h_1 \\ t_1 & a+b+2\,a\,b \\ t_2 & a+b+2\,a\,b \end{pmatrix}\right\}$$

It is possible, nevertheless, to slightly complicate the β calculus to make the above identity hold (in fact, this amounts to "simplfying it less" from its origin in [REF]):

In this alternate scheme, we use variables c_x instead of their exponentials T_x. We have different "R-matrices" given by:

$$R^+_{xy} = \begin{array}{c|cc} 1 & h_x & h_y \\ \hline t_x & 0 & c_x^{-1}(e^{c_x}-1) \\ t_y & 0 & 0 \end{array} \qquad R^-_{xy} = \begin{array}{c|cc} 1 & h_x & h_y \\ \hline t_x & 0 & c_x^{-1}(e^{-c_x}-1) \\ t_y & 0 & 0 \end{array}$$

The operations are the same as before except that the "column norm" is now $\langle \mu \rangle = \sum_i c_i \alpha_i$. This fixes the "bug" above and makes all the axioms of a Hopf meta-bicrossed product hold. The verifications are left to the reader.

4.2 Theorem 2