Following

$$\#_1\left(\begin{array}{c}
\circ \\
\circ
\end{array}\right) \cong \mathbb{Z}$$

(virtual tangle)

Q. Is there a bijection between w-links and "Wirtinger groups"?

Kamada: A characterisation of w-braids that have the same w-link closure (i.e., Kamada has a Markov thin for w-braids)

From paper:

Theorem 2. [2] There is a representation \( \psi \) of \( VB_n \) in \( Aut(F_{n+1}) \), \( F_{n+1} = \langle x_1, x_2, \ldots, x_n, y \rangle \) which is defined by the following actions on the generators of \( VB_n \):

\[
\psi(\sigma_i) : \begin{cases} 
    x_i \mapsto x_i x_{i+1} x_i^{-1} , \\
    x_{i+1} \mapsto x_i , \\
    x_l \mapsto x_l , \ l \neq i, i+1 ; \\
    y \mapsto y ,
\end{cases}
\]

\[
\psi(\rho_i) : \begin{cases} 
    x_i \mapsto y x_{i+1} y^{-1} , \\
    x_{i+1} \mapsto y^{-1} x_i y , \\
    x_l \mapsto x_l , \ l \neq i, i+1 , \\
    y \mapsto y ,
\end{cases}
\]

for all \( i = 1, 2, \ldots, n-1 \).

\[
\begin{array}{ccc}
F_{n+1} & \xrightarrow{\text{"forget y"}} & F_n \\
\downarrow V_{B_n} & & \downarrow W_{B_n} \\
F_{n+1} & \xrightarrow{\text{Commutes}} & F_n
\end{array}
\]
So if $G_{F_{n+1}}$ is faithful, then there is a faithful representation of $V_{B_n}$ in $W_{B_{n+1}}$.

Also by "Markov", get an "extended fundamental group" invariant of v-brads, $G_v(vk)$ does not detect the kishino knot.

[Though adding a "peripheral system", kishino is detected].

Claim $G_v(vk)/\langle y \rangle \cong \pi_1(vk)$

Wada representations: [of $uB_n$]

$x_i \circ_i = u(x_i, x_{i+1})$, $u, v \in F_2$

$x_{i+1} \circ_i = v(x_i, x_{i+1})$

$x_j \circ_i = x_j$ otherwise.

Prop (Wada) There are at least 7 type of Wada reps.

Jto (2011) There are exactly 7 such reps, up to involution $x_i \rightarrow x_i$, $x_i \rightarrow x_i$ of $F_n$ or/and $B_n$. 