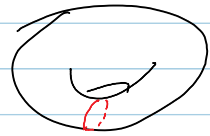
Chaika@SC: Interval exchange transformations

January-16-12
4:09 PM

$R_\alpha : S^1 \circlearrowleft \qquad R_\alpha(e^{2\pi i x}) = e^{2\pi i (x+\alpha)}$

$[0,1] \circlearrowleft \qquad x \longmapsto (x+\alpha) \bmod 1$

Also occurs as the "return to a circle" behavior of flows on the torus:



A similar thing appears for billiards.

$[0,1] \xrightarrow{\tau} \{1,2\}^{\mathbb{Z}}$ by $a_i = \begin{cases} 1 & \text{if } R_\alpha^i \in [0,\alpha) \\ 2 & \text{otherwise.} \end{cases}$

$B_n = \{(a_1 \dots a_n) : \exists x \text{ s.t. } \tau_i(x) = a_i\}$

Then $|B_n| \le n+1$

TFAE:

1. $\{R_\alpha^i(x)\}$ is dense.
2. The slope of the torus line is $\notin \mathbb{Q}$.
3. α is irrational.
4. $|B_n| = n+1$
5. $\frac{1}{N} \sum_{i=0}^{N-1} \chi_{[a,b]}(R_\alpha^i(x)) = b-a$

Thm (Kurzweil, 1950's)

1. Fix $b_i \downarrow$, $\sum b_i = \infty$ Then for a.e. (x,y),

$|R_\alpha^i x - y| < b_i$ for ∞-many i

2. For a.e. α $\exists b_i \downarrow$ $\sum b_i = \infty$ s.t.

for a.e. (x,y) $|R_\alpha^i x - y| < b_i$ only finitely many times

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Thm (Classical) $b_i \downarrow$, $\sum b_i = \infty$, a.e. α

$$|R_\alpha^i(x) - x| < b_i \text{ for } \infty \text{ many } i.$$

Interval exchange transformations.

— are determined by a perutation and the lengths. The lengths make a point in the simplex.

Show up as "straight line flows on translation surfaces", and as "billiards in a rational polygon".

An IET defines a subset of $\{1..d\}^{\mathbb{Z}}$ & get $|B_n| \leq n(d-1)+1$

An IET T may be ergodic w/o uniquely ergodic.

Thm (joint w/ advisor) If T is μ-ergodic, then $\mu\times\mu$ a.e. $(x,y)\in I^2$,

$$|T^n x - y| < \varepsilon/n \quad \infty\text{-often}.$$ 4:45

various other versions of the R_α Theorems in the IET case.