A Bit on Maxwell’s Equations

Prerequisites.

- Poincaré’s Lemma, which says that on \( \mathbb{R}^n \), every closed form is exact. That is, if \( d\omega = 0 \), then there exists \( \eta \) with \( d\eta = \omega \).
- Integration by parts:
  \[
  \int \omega \wedge d\eta = -(-1)^{\deg \omega} \frac{\partial}{\partial t} \int (d\omega) \wedge \eta
  \]
  on domains that have no boundary.
- The Hodge star operator \( \star \) which satisfies
  \[
  \omega \wedge \star \eta = \langle \omega, \eta \rangle dx_1 \cdots dx_n
  \]
  whenever \( \omega \) and \( \eta \) are of the same degree.
- The simplest least action principle: the extremes of
  \( q \mapsto \int_a^b \left( \frac{1}{2}m\dot{q}^2(t) - V(q(t)) \right) dt \)
  occur when \( m\ddot{q} = -V'(q(t)) \). That is, when \( F = ma \).

The Action Principle. The Vector Field is a compactly supported 1-form \( A \) on \( \mathbb{R}^4 \) which extremizes the action

\[
S_J(A) := \int_{\mathbb{R}^4} \frac{1}{2}||dA||^2 dt dx dy dz + J \wedge A
\]

where the 3-form \( J \) is the charge-current.

The Euler-Lagrange Equations in this case are \( d \star dA = J \), meaning that there’s no hope for a solution unless \( dJ = 0 \), and that we might as well (think Poincaré’s Lemma!) change variables to \( F := dA \). We thus get

\[
dJ = 0 \quad dF = 0 \quad d \star F = J
\]

These are the Maxwell equations! Indeed, writing \( F = (E_x dx dt + E_y dy dt + E_z dz dt) + (B_x dy dz + B_y dz dx + B_z dx dy) \) and \( J = \rho dx dy dz - j_x dy dz dt - j_y dz dx dt - j_z dx dy dt \), we find:

\[
\begin{align*}
dJ &= 0 \implies \frac{\partial \rho}{\partial t} + \text{div} j = 0 & \text{“conservation of charge”} \\
dF &= 0 \implies \text{div } B = 0 & \text{“no magnetic monopoles”} \\
& \quad \text{curl } E = -\frac{\partial B}{\partial t} & \text{that’s how generators work!} \\
d \star F &= J \implies \text{curl } E = -\rho & \text{“electrostatics”} \\
& \quad \text{curl } B = -\frac{\partial E}{\partial t} + j & \text{that’s how electromagnets work!}
\end{align*}
\]

Exercise. Use the Lorentz metric to fix the sign errors.

Exercise. Use pullbacks along Lorentz transformations to figure out how \( E \) and \( B \) (and \( j \) and \( \rho \)) appear to moving observers.

Exercise. With \( ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \) use \( S = mc \int_{c_1}^{c_2} (ds + eA) \) to derive Feynman’s “law of motion” and “force law”.

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