A Bit on Maxwell's Equations

Prerequisites.

- Poincaré's Lemma, which says that on $\mathbb{R}^n$, every closed form is exact. That is, if $d\omega = 0$, then there exists η with $d\eta = \omega$.
- Integration by parts: $\int \omega \wedge d\eta = -(-1)^{\deg \omega} \int (d\omega) \wedge \eta$ on domains that have no boundary.
- The Hodge star operator $\star$ which satisfies $\omega \wedge \star\eta = \langle \omega, \eta \rangle dx_1 \cdots dx_n$ whenever ω and η are of the same degree.
- The simplesest least action principle: the extremes of $q \mapsto \int_a^b \left(\frac{1}{2}m\dot{q}^2(t) - V(q(t))\right) dt$ occur when $m\ddot{q} = -V'(q(t))$. That is, when $F = ma$.

Table 18–1 Classical Physics

Maxwell's equations	
I. $\nabla \cdot \boldsymbol{E} = \dfrac{\rho}{\epsilon_0}$	(Flux of $\boldsymbol{E}$ through a closed surface) = (Charge inside)/ϵ_0
II. $\nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t}$	(Line integral of $\boldsymbol{E}$ around a loop) $= -\dfrac{d}{dt}$ (Flux of $\boldsymbol{B}$ through the loop)
III. $\nabla \cdot \boldsymbol{B} = 0$	(Flux of $\boldsymbol{B}$ through a closed surface) = 0
IV. $c^2\nabla \times \boldsymbol{B} = \dfrac{\boldsymbol{j}}{\epsilon_0} + \dfrac{\partial \boldsymbol{E}}{\partial t}$	c^2(Integral of $\boldsymbol{B}$ around a loop) = (Current through the loop)/ϵ_0 $+ \dfrac{\partial}{\partial t}$ (Flux of $\boldsymbol{E}$ through the loop)
Conservation of charge $\nabla \cdot \boldsymbol{j} = -\dfrac{\partial \rho}{\partial t}$	(Flux of current through a closed surface) $= -\dfrac{\partial}{\partial t}$ (Charge inside)
Force law $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$	
Law of motion $\dfrac{d}{dt}(\boldsymbol{p}) = \boldsymbol{F}$, where $\boldsymbol{p} = \dfrac{m\boldsymbol{v}}{\sqrt{1 - v^2/c^2}}$	(Newton's law, with Einstein's modification)
Gravitation $\boldsymbol{F} = -G\dfrac{m_1 m_2}{r^2}\boldsymbol{e}_r$	

The Feynman Lectures on Physics vol. II, page 18-2

The Action Principle. The *Vector Field* is a compactly supported 1-form A on $\mathbb{R}^4$ which extremizes the *action*

$$\boxed{S_J(A) := \int_{\mathbb{R}^4} \frac{1}{2}||dA||^2 dtdxdydz + J \wedge A}$$

where the 3-form J is the *charge-current*.

The Euler-Lagrange Equations in this case are $d \star dA = J$, meaning that there's no hope for a solution unless $dJ = 0$, and that we might as well (think Poincaré's Lemma!) change variables to $F := dA$. We thus get

$$\boxed{dJ = 0 \qquad dF = 0 \qquad d \star F = J}$$

These are the Maxwell equations! Indeed, writing $F = (E_x dxdt + E_y dydt + E_z dzdt) + (B_x dydz + B_y dzdx + B_z dxdy)$ and $J = \rho dxdydz - j_x dydzdt - j_y dzdxdt - j_z dxdydt$, we find:

$$\begin{array}{|rcl|}
\hline
dJ = 0 \Longrightarrow & \dfrac{\partial \rho}{\partial t} + \operatorname{div} j = 0 & \text{“conservation of charge”} \\
dF = 0 \Longrightarrow & \operatorname{div} B = 0 & \text{“no magnetic monopoles”} \\
 & \operatorname{curl} E = -\dfrac{\partial B}{\partial t} & \text{that's how generators work!} \\
d * F = J \Longrightarrow & \operatorname{div} E = -\rho & \text{“electrostatics”} \\
 & \operatorname{curl} B = -\dfrac{\partial E}{\partial t} + j & \text{that's how electromagnets work!} \\
\hline
\end{array}$$

Exercise. Use the Lorentz metric to fix the sign errors.

Exercise. Use pullbacks along Lorentz transformations to figure out how E and B (and j and ρ) appear to moving observers.

Exercise. With $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ use $S = mc\int_{e_1}^{e^2}(ds + eA)$ to derive Feynman's “law of motion” and “force law”.