The Blob complex: A pairing between n-manifolds and n-categories, producing a chain complex.

\[ Z(M; E) \xrightarrow{H_0} \mathbb{Z} \]

\[ \mathcal{P}_* (M; E) \xrightarrow{n=1} \text{H} \rightarrow \text{H} \rightarrow \cdots \]

\[ \mathcal{C} = K[t] \]

\[ C_*(\Delta^\infty M) \]

What is a disk-like n-category?

* A collection of functors

\[ \mathcal{G}_k : [k\text{-balls}] \rightarrow \text{Set} \text{ for } k = 0, 1, \ldots, n \]

homeomorphisms

\[ \mathcal{G}_k(X) = \text{"} k\text{-morphisms of shape } X \text{"} \]

* Restriction maps: Given \( Y^k \subset \partial X^k \),

[maps] \( \mathcal{G}_k(X) \rightarrow \mathcal{G}_{k-1}(Y) \)
Composition: maps give

\[ E_k(X_1) \times E_k(X_2) \to E_k(X_1 \cup X_2) \]

Isn't this the proof that the subject of n-categories is stupid to start with?

These data must satisfy:

1. Gluing & homeomorphisms are compatible.
2. Gluing is associative.
3. Isotopy invariance at level \( n \).

Examples. \( n=1 \) any \(*\)-category \( C \) gives a disklike \((-\)category:

\[ E_0(\cdot) = \text{Obj}(C) \quad \text{objects of } C \]

\[ E_1(\cdot) = \text{Mor}(\cdot) \quad \Uparrow \text{isotopy} \quad \Uparrow \text{composition} \]

\[ \text{morphisms of } C \]

\( n=2 \) example:

\[ E_k(X) = \# \text{ codimension }-1 \text{ submanifolds of } X \]

except

\[ E_2(X) = \left( \text{cyl } \cup \delta \right) / \text{isotopy} / \Omega = \delta \]

This is the Temperley-Lieb category.
Example:

\[ \mathcal{E}_0 \{ \emptyset \} = \{ \} \]

\[ \mathcal{E}_1 \{ \sim \} = \{ \text{twisted surfaces} \} \]

\[ \mathcal{E}_2 \{ \circ \} \]

\[ \mathcal{E}_3 \{ \bigcirc \} = \left\{ \text{contact structures} \right\} \text{ on } \mathbb{B} \]

\( n = 4 \text{ example: skipped, but one exists, based on Khovanov homology } \)

**Arbitrary** \( n := \)

\[ \mathcal{E}_k (X) = \text{maps} (X \rightarrow T) \]

with

\[ \mathcal{E}_n (X) = \text{maps} (X \rightarrow T) / \text{homotopy} \]

for some topological space \( T \).

**Arbitrary** \( n := \) given a commutative ring \( R \) or even commutative monoid,

\[ \mathcal{E}_k (X) = \left\{ X \right\} \]

\[ \mathcal{E}_n (X) = \left\{ \frac{X}{\sim} \right\} \text{ } / \text{ } \left\{ \left\{ X \right\} / \text{rel } \mathbb{B}, \mathbb{C}, \mathbb{R} \right\} \]
Let extend to \( n \)-manifolds:

\[
\mathcal{E}(W) = \bigg( \bigg( \prod_{i} \mathcal{E}(X_i) \bigg)_{\text{fib}} \bigg)_{\text{Rel}} \bigg( \bigcup_{i} X_i \bigg) \bigg)_{\text{Rel}}
\]

\( \text{Rel}: = \bigcirc = \bigcirc \bigg( \mathcal{E}(M) \bigg)
\]

= colim over the poset of ball decompositions, under coarsening.

Anti-climatically,

\[
\mathcal{Q}_{\ast}(M, \mathcal{E}) = \text{hocolim \ (same)}
\]

\[
\mathcal{Q}_{0}(M, \mathcal{E}) = \bigg( \bigg( \bigg( \bigg)_{\mathcal{E}(M) \times \mathcal{E}(X)} \bigg)_{\text{elem}} \bigg)_{\text{in } \mathcal{D}(M)}
\]

\[
\mathcal{Q}_{1}(M, \mathcal{E}) = \bigg( \bigg( \bigg( \bigg)_{\mathcal{E}(M) \times \mathcal{E}(X)} \bigg)_{\text{elem}} \bigg)_{\text{in } \mathcal{D}(M)}
\]