The Blob Complex: A pairing between n-manifolds and n-categories, producing a chain complex.

$$
\begin{tikzcd}
 & Z(M;\mathcal{C}) & \\
 & \mathcal{B}_*(M,\mathcal{C}) \arrow[u, "H_0"] \arrow[r, "n=1", "M=S^1"'] \arrow[dl, "\mathcal{C}=k[t]"'] & HH_*(\mathcal{C}) \\
C_*(\Delta^\infty M) & &
\end{tikzcd}
$$

The usual TQFT Hilbert Space

manifold

n-category

What is a disk-like n-category?

* A collection of functors

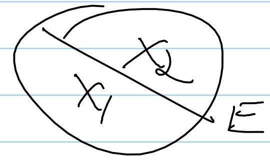
$$\mathcal{C}_k : \left\{ \begin{array}{c} k\text{-balls} \\ \circlearrowleft \end{array} \right\} \longrightarrow \mathrm{Set} \quad \text{for } k=0,\dots,n$$

homeomorphisms

$\mathcal{C}_k(X) =$ " k-morphisms of shape X "

* Restriction maps: Given $Y^{k-1} \subset \partial X^k$, maps $\partial : \mathcal{C}_k(X) \longrightarrow \mathcal{C}_{k-1}(Y)$

maps $\partial : \mathcal{C}_k(X) \to \mathcal{C}_{k-1}(\partial X)$

* Composition: maps given 

$$\mathcal{C}_k(X_1) \times_{\mathcal{C}_{k-1}(E)} \mathcal{C}_k(X_2) \longrightarrow \mathcal{C}_k(X_1 \cup X_2)$$

Isn't this the proof that the subject of n-categories is stupid to start with?

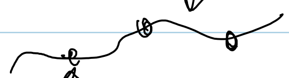
These data must satisfy:

1. Gluing & homeomorphisms are compatible.
2. Gluing is associative.
3. isotopy invariance at level n.

Examples. $n=1$ any $*$-category C gives a disklike 1-category:

$$\mathcal{C}_0(\cdot) = \mathrm{Obj}(C)$$

$$\mathcal{C}_1(\sim) = \{ \quad \} / \text{isotopy, composition}$$



objects of C

morphisms of C

$n=2$ example:

$$\mathcal{C}_k(X) = \left\{ \begin{array}{c} \text{codimension-1} \\ \text{submanifolds} \\ \text{of } X \end{array} \right\}$$

except

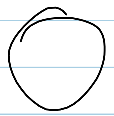
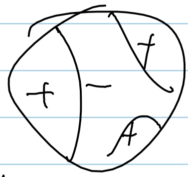
$$\mathcal{C}_2(X) = \mathbb{C}\{\text{same}\} / {}^{\text{isotopy}}/_{\bigcirc = \delta}$$

This is the Temperley-Lieb Category.

n=3 example:

$$\mathcal{C}_0(\cdot) = \{\cdot\}$$

$\mathcal{C}_1(\sim) = \{$  $\}$

$\mathcal{C}_2($  $) = \{$  $\}$

$\mathcal{C}_3($  $) = \left\{\begin{array}{l}\text{tight contact structures} \\ \text{on } B\end{array}\right\}$

(may "point in" or "out" on bndry)

n=4 example ... skipped, but one exists, based on Khovanov homology.

Arbitrary n:

$$\mathcal{C}_k(X) = \text{maps}(X \to T)$$

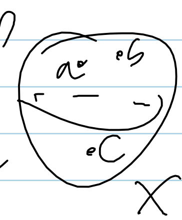
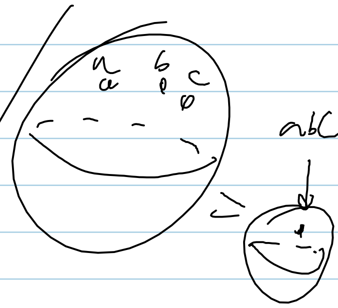
w/

$$\mathcal{C}_n(X) = \text{maps}(X \longrightarrow T)/\text{homotopy}$$

for some topological space T.

Arbitrary n: given a commutative ring R or even commutative monoid.

$$\mathcal{C}_k(X) = \{X\} \quad k<n$$

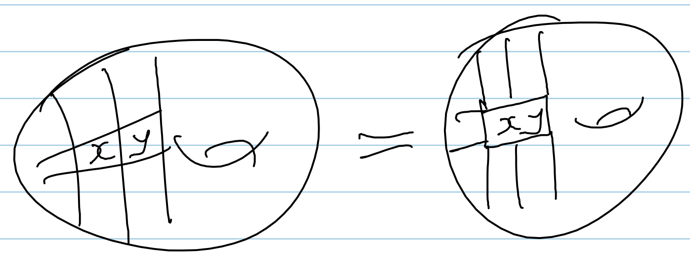
$\mathcal{C}_n(X) = \{$  $\}$ / 

a,b,c ∈ R

Let extend to n-manifolds:

$$\underrightarrow{\mathcal{C}}(W) = \coprod_{\substack{\text{all decompositions}\\ \text{of } W \text{ into balls}\\ W = \cup X_i}} \underbrace{\prod}_{\text{fibered product}} \mathcal{C}(X_i) \Big/ \text{Rel}$$

any n-manifold

Rel: = 

$= \operatorname{Colim}$ over the poset of ball decompositions, under coarsening. $\mathcal{D}(M)$

Anti-climatically,

$$\mathcal{D}_*(M, \mathcal{C}) = \text{hocolim (same)}$$

$$\mathcal{D}_0(M, \mathcal{C}) = \bigoplus_{\mathcal{D}(M)} \bigotimes_{x \in b} \mathcal{C}(X)$$

$$\mathcal{D}_1(M, \mathcal{C}) = \bigoplus_{\substack{\text{elementary}\\ \text{arrows}\\ \text{in } \mathcal{D}(M)}} \bigotimes_{x \in b} \mathcal{C}(X)$$

$\vdots$