Projective Modules

\[ \exists B \xrightarrow{f} P \rightarrow A \]

\[ \mathcal{P} \xrightarrow{\cong} F \quad \text{True if } \mathcal{P} \text{ is free} \]

\[ \mathcal{B} \rightarrow A \rightarrow 0 \]

**Claim:** If \( \mathcal{P} \xrightarrow{\cong} F \) then \( F = \mathcal{P} \oplus (\ker T) \).

Q. What is the fundamental reason why this is significant?

**Claim:** If \( 0 \rightarrow P \rightarrow Q \rightarrow R \rightarrow 0 \) is exact,
then \( Q = P \oplus R \).

By projectivity of \( R \), \( Q \xrightarrow{\cong} R \rightarrow 0 \)

The sequence splits:

\[ 0 \rightarrow P \rightarrow Q \xleftarrow{\cong} R \rightarrow 0 \]

We've only used the projectivity of \( R \) here.
In fact, we've only used that every surjection on \( B \) has a section. **Converse?**

Claim: If every surjection on \( P \) has a section, then \( P \) is projective.

\[
\begin{array}{c}
\text{P} \\
\downarrow \\
\text{B} \rightarrow \text{A} \rightarrow \text{C}
\end{array}
\]