

Relevant Sources

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From Stonehenge to Witten Skipping all the Details

Oporto Meeting on Geometry, Topology and Physics, July 2004
Dror Bar-Natan, University of Toronto

It is well known that when the sun rises on midsummer's morning over the "Heel Stone" at Stonehenge, its first rays shine right through the open arms of the horseshoe arrangement. These astronomical lineups, one of the pillars of modern thought, are much older than the famed Gaussian linking number of two knots.

$(D, K)_H := \text{(The signed Stonehenge)} : \text{pairing of } D \text{ and } K$

$D = \text{ } \quad K = \text{ } \quad \Rightarrow \text{ }$

The Gaussian linking number $B_K(D) = \frac{1}{2} \sum_{\text{displinks}} (\text{signs})$

Carl Friedrich Gauss

Thus we consider the generating function of all stellar coincidences:

$Z(K) := \lim_{N \rightarrow \infty} \sum_{\text{all } D \text{ with } N \text{ edges}} \frac{1}{2^N} \delta^{(N)}((D, K)_H D) \in \mathcal{A}(\mathbb{C})$

Theorem. Modulo Relations, $Z(K)$ is a knot invariant!

When deforming, catastrophes occur when:
A plane moves over an intersection point - An intersection line cuts through the knot - Solution: Impose HKX.
 $\text{H} = \text{H} \times \text{H}$ (see below) - Similar argument (similar argument shown here)

The HKX Relation

It all is perturbative Chern-Simons-Witten theory:

$\int_{\text{D-Feynman}} D A \log(A) \exp \left[\frac{i k}{4 \pi} \int_{\text{S}} (A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \right]$

$\rightarrow \sum_{\text{D-Feynman diagrams}} W_g(D) \sum_{\text{D-Feynman diagrams}} D \sum_{\text{D-Feynman diagrams}}$

Shing-Tung Yau

James H. Simons

This handout is at <http://www.math.toronto.edu/~drorbn/Talks/Oporto-0407/>

More at <http://www.math.toronto.edu/~drorbn/Talks/Oporto-0407/>

From Stonehenge to Witten – Some Further Details

Oporto Meeting on Geometry, Topology and Physics, July 2004
Dror Bar-Natan, University of Toronto

We're generating function of all stellar coincidences:

$Z(K) := \lim_{N \rightarrow \infty} \sum_{\text{all } D \text{ with } N \text{ edges}} \frac{1}{2^N} \delta^{(N)}((D, K)_H D) \in \mathcal{A}(\mathbb{C})$

$(D, K)_H := \text{(The signed Stonehenge)} : \text{pairing of } D \text{ and } K$

$D = \text{ } \quad K = \text{ } \quad \Rightarrow \text{ }$

Theorem. Modulo Relations, $Z(K)$ is a knot invariant!

$\int_{\text{D-Feynman}} D A \log(A) \exp \left[\frac{i k}{4 \pi} \int_{\text{S}} (A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \right] \rightarrow \sum_{\text{D-Feynman diagrams}} W_g(D) \sum_{\text{D-Feynman diagrams}} D \sum_{\text{D-Feynman diagrams}}$

Related to Lie algebras

More precisely, let $V = \{V_i\}_{i=1}^n$ be a Lie algebra with an orthonormal basis, and let $R = \{r_{ij}\}$ be a representation. Set

$f_{ijk} := \langle [V_i, V_j], V_k \rangle$

and then

$W_{F,R} := \sum_{\text{D-Feynman diagrams}} f_{ijk} r_{ij}^a r_{jk}^b r_{ki}^c$

Plurality and the Yang-Baxter equation

Kauffman's bracket and the Jones polynomial!

$\langle \text{---} \rangle = \langle \text{---} \rangle - q \langle \text{---} \rangle$

$\langle \text{---} \rangle = \langle \text{---} \rangle - q^{-1} \langle \text{---} \rangle$

$\langle \text{---} \rangle = \langle \text{---} \rangle - q \langle \text{---} \rangle$

$\langle \text{---} \rangle = \langle \text{---} \rangle - q^{-1} \langle \text{---} \rangle$

"God created the knots, all else in topology is the work of man." This handout is at <http://www.math.toronto.edu/~drorbn/Talks/Oporto-0407/>

More at <http://www.math.toronto.edu/~drorbn/Talks/Oporto-0407/>

Knotted Trivalent Graphs, Tetrahedra and Associates

HUJI Topology and Geometry Seminar, November 16, 2000
Dror Bar-Natan

Goal: $Z(\text{knots}) \rightarrow (\text{chord diagrams})/4T$ so that

The Milne bracket law:

Modulo the relation(s):

Claim. With $\Delta := Z(\Delta)$, the above relation becomes equivalent to the Drinfel'd pentagon of the theory of quasi Hopf algebras.

Proof.

Extend to Knotted Trivalent Graphs (KTG's):

Need a new relation:

Easy, powerful moves:

Using moves, TKG is generated by ribbon twists and the tetrahedron Δ :

Further directions:

1. Relations with perturbative Chern-Simons theory.
2. Relations with the theory of 6j symbols
3. Relations with the Turay-Viro invariants
4. Can this be used to prove the Witten asymptotics conjecture?
5. Does this extend/improve Drinfel'd's theory of associates?

This handout is at <http://www.ma.huji.ac.il/~drorbn/Talks/HUJI-001116/>

More at <http://www.math.toronto.edu/~drorbn/Talks/HUJI-001116/>

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Convolutions on Lie Groups and Lie Algebras and Ribbon 2-Knots

Dror Bar-Natan, Bonn August 2009, <http://www.math.toronto.edu/~drorbn/Talks/Bonn-0908>

The Bigger Picture... What are w-Brackets? The Orbits Method
Group-Algebraic statement
Unitary statement
Free Lie statement
Algebraic statement
Diagrammatic statement
Homological statement
Lie-Tressean statement
Knots
Vergne
Torrecilla
A ribbon 2-knot is a surface S embedded in \mathbb{R}^4 that bounds an immersed handlebody B , with only "ribbon singularities": a ribbon singularity is a disk D of traverse double points, whose preimages in B are a disk D_0 in the interior of B and a disk D_2 with $D_2 \cap \partial B = \partial D_0$, modulo isotopies of S alone.

Homomorphic expansion for a filtered algebraic structure K :
 $\text{ops} \leq K = K_0 \supseteq K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots$
 $\text{ops} \leq \text{gr } K := K_0/K_1 \oplus K_1/K_2 \oplus K_2/K_3 \oplus K_3/K_4 \oplus \dots$

An expansion is a filtration respecting $\text{gr } K$ that "covers" the identity on $\text{gr } K$. A homomorphic expansion is an expansion that all relevant "extra" operations.

Filtered algebraic structures are cheap and plenty. In any K , allow formal linear combinations. Let K_\bullet be the ideal generated by differences (the "augmentation ideal"), and let $K_n := \langle \{K_\bullet\}^n \rangle$ (using all available "products").

"An Algebraic Structure"

• Has kinds, objects, operations, and maybe constants.
• Perhaps subject to some axioms.
• We always allow formal linear combinations.

Example: Pure Braids. P_{R_n} is generated by x_{ij} , "strand i goes around strand j once", modulo "Reidemeister moves". $A_n := \text{gr } P_{R_n}$ is generated by $t_{ij} = x_{ij} - 1$, modulo the 4T relations $t_{ij}t_{kl} + t_{kl}t_{ij} = 0$ (and some lesser ones too). Much depends on A_n , including the Drinfel'd theory of associators.

Our case: $K = \frac{\mathbb{Z}}{2} \otimes \text{high algebra}$
 $\text{ops} \leq K = \frac{\mathbb{Z}}{2} \otimes \text{Lie algebra } \mathfrak{g}$
 $\text{ops} \leq \text{gr } K = \frac{\mathbb{Z}}{2} \otimes \text{Lie algebra } \mathfrak{g}$
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Disclaimer: Ribbon edges carry topology in the work of others. Legend: $\text{PA} = \text{Planar Algebra}$

God created the knots, all else is topology in the work of mortals.

What are w-Brackets?
 $\text{knots} = \text{PA} \langle \text{R23}, \text{OC}, \text{VR123}, \text{D}, \text{OC}, \text{R4} \rangle$
 $\text{w-tangles} = \text{PA} \langle \text{w-generators}, \text{w-relations}, \text{w-unitary operations} \rangle$
 $\text{w-T} = \{ \text{w-tangles} \} = \text{PA} \langle \text{w-generators}, \text{w-relations}, \text{w-unitary operations} \rangle$

The w-generators
 Braiding relation
 Crossing
 Cap Was
 Virtual crossing
 Move
 Vertices
 w-annulus
 w-disk

Challenge: Do the Reidemeister!

Just for fun:
 $K = \{ \text{the set of all knots} \}$
 $K = \{ \text{b/w 2D gradations of reality} \}$
 $K = \{ \text{K}_1, K_1 \cup K_2, K_1 \cup K_2 \cup K_3, K_1 \cup K_2 \cup K_3 \cup K_4, \dots \}$
 $K = \{ \text{K}_1, K_1 \cup K_2, K_1 \cup K_2 \cup K_3, K_1 \cup K_2 \cup K_3 \cup K_4, \dots \}$
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Convolutions
 $\text{R23} = \text{PA} \langle \text{R23}, \text{OC}, \text{VR123}, \text{D}, \text{OC}, \text{R4} \rangle$
 $\text{VR123} = \text{PA} \langle \text{VR123}, \text{D}, \text{OC}, \text{R4} \rangle$
 $\text{D} = \text{PA} \langle \text{D}, \text{OC} \rangle$
 $\text{OC} = \text{PA} \langle \text{OC} \rangle$
 $\text{R4} = \text{PA} \langle \text{R4} \rangle$
 $\text{VR123} = \text{PA} \langle \text{VR123} \rangle$
 $\text{R23} = \text{PA} \langle \text{R23} \rangle$
 $\text{OC} = \text{PA} \langle \text{OC} \rangle$
 $\text{R4} = \text{PA} \langle \text{R4} \rangle$

Diagrammatic statement: Let $R = \exp^{14} \in \mathcal{A}^w(\mathbb{T})$. There exist $w \in \mathcal{A}^w(\mathbb{T})$ and $V \in \mathcal{A}^w(\mathbb{T})$ so that

Diagrammatic to Algebraic: With (x_i) and (φ^i) dual bases of \mathfrak{g} and with $x_i \cdot x_j = \sum b_{ij}^k x_k$, we have $\mathcal{A}^w = \mathcal{U}$ via

Algebraic statement: With $Ig := g^1 \times g_2$, with $c : \mathcal{U}(Ig) \rightarrow \mathcal{U}(Ig)/\mathcal{U}(Ig)^\perp = S(g^2)$ the obvious projection, with S the exponential of $\mathcal{U}(Ig)$, but not U , with the automorphism of $\mathcal{U}(Ig)$ induced by flipping \mathfrak{g} with φ^i , with $\varphi^i \circ \varphi^j = \varphi^{i+j}$ the identity element and with $R = c^{-1} \in \mathcal{U}(Ig) \otimes \mathcal{U}(Ig)$ there exist $w \in S(g)$ and $V \in \mathcal{U}(Ig)^{\otimes 2}$ so that

Diagrams: (1) $\text{VR123} = R^{13} \text{VR23}^2$ in $\mathcal{U}(Ig)^{\otimes 2} \otimes \mathcal{U}(Ig)$
(2) $\text{SW1-V} = 1$
(3) $\text{V} \circ_{wxy} = w_{xy}$

Unitary statement: There exists $w \in \text{Fun}(\mathfrak{g})^G$ and an (infinite order) tangential differential operator V defined on $\text{Fun}(\mathfrak{g})^G$ so that

(1) $\text{V} \circ_{wxy} = \tilde{c} \tilde{x} \tilde{y} V$ (allowing $\mathcal{U}(Ig)$ -valued functions)
(2) $\text{VV} = 1$
(3) $\text{V} \circ_{wxy} = w_{xy}$

Group-Algebra statement: There exists $w^U \in \text{Fun}(\mathfrak{g})^G$ so that for every $\phi, \psi \in \text{Fun}(\mathfrak{g})^G$ (with small support), the following holds in $\mathcal{U}(g)$:

Convolutional statement (Kashiwara-Vergne): Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately: let G be a finite dimensional Lie group and let \mathfrak{g} be its Lie algebra, let $\varphi : \mathfrak{g} \rightarrow \mathbb{R}$ be the Jacobian of the exponential map $\exp : \mathfrak{g} \rightarrow G$, and let $\Phi : \text{Fun}(\mathfrak{g}) \rightarrow \text{Fun}(G)$ be given by $\Phi(\varphi) = \varphi \circ \exp$. If $f, g \in \text{Fun}(G)$ are Ad-invariant and supported near the identity, then $\Phi(f) * \Phi(g) = \Phi(f * g)$.

Video and more at <http://www.math.toronto.edu/~drorbn/Talks/Bonn-0908/>

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Convolutions on Lie Groups and Lie Algebras and Ribbon 2-Knots, Page 2

From wTT to \mathcal{A}^w : $gr_m wTT := (m\text{-cubes})/\{(m-1)\text{-cubes}\}$

w-Jacobi diagrams and $\mathcal{A}^w(\mathbb{T})$: $\mathcal{A}^w(\mathbb{T}) \cong \mathcal{A}^w(\mathbb{T})$ is

Diagrammatic statement: Let $R = \exp^{14} \in \mathcal{A}^w(\mathbb{T})$. There exist $w \in \mathcal{A}^w(\mathbb{T})$ and $V \in \mathcal{A}^w(\mathbb{T})$ so that

Diagrammatic to Algebraic: With (x_i) and (φ^i) dual bases of \mathfrak{g} and with $x_i \cdot x_j = \sum b_{ij}^k x_k$, we have $\mathcal{A}^w = \mathcal{U}$ via

Algebraic statement: With $Ig := g^1 \times g_2$, with $c : \mathcal{U}(Ig) \rightarrow \mathcal{U}(Ig)/\mathcal{U}(Ig)^\perp = S(g^2)$ the obvious projection, with S the exponential of $\mathcal{U}(Ig)$, but not U , with the automorphism of $\mathcal{U}(Ig)$ induced by flipping \mathfrak{g} with φ^i , with $\varphi^i \circ \varphi^j = \varphi^{i+j}$ the identity element and with $R = c^{-1} \in \mathcal{U}(Ig) \otimes \mathcal{U}(Ig)$ there exist $w \in S(g)$ and $V \in \mathcal{U}(Ig)^{\otimes 2}$ so that

Diagrams: (1) $\text{VR123} = R^{13} \text{VR23}^2$ in $\mathcal{U}(Ig)^{\otimes 2} \otimes \mathcal{U}(Ig)$
(2) $\text{SW1-V} = 1$
(3) $\text{V} \circ_{wxy} = w_{xy}$

Unitary \rightarrow Algebraic: The key is to interpret $\mathcal{U}(Ig)$ as tangent spaces. $\mathfrak{g} \times \mathfrak{g}^*$ becomes a multiplication operator:
 $\mathfrak{g} \times \mathfrak{g}^*$ becomes a tangential derivation, in the direction of the action of $\text{ad } x: (x \varphi)(y) := \varphi([x, y])$.
 $\mathfrak{g} \times \mathfrak{g}^*$ becomes $\mathcal{U}(Ig) \otimes \mathcal{U}(Ig) = S(\{x, y\})$ is "the constant term".

Unitary \rightarrow Group-Algebra: The key is to interpret $\mathcal{U}(Ig)$ as tangent spaces. $\mathfrak{g} \times \mathfrak{g}^*$ becomes a multiplication operator:
 $\mathfrak{g} \times \mathfrak{g}^*$ becomes a tangential derivation, in the direction of the action of $\text{ad } x: (x \varphi)(y) := \varphi([x, y])$.
 $\mathfrak{g} \times \mathfrak{g}^*$ becomes $\mathcal{U}(Ig) \otimes \mathcal{U}(Ig) = S(\{x, y\})$ is "the constant term".

Convolutions and Group-Algebra (ignoring all Jacobians): If \mathfrak{g} is finite, A is a Lie algebra, $\tau : G \rightarrow A$ is multiplicative then $\text{Fun}(G, *) \cong (\mathcal{A}^w)$ via $L : f \mapsto \int f(a) \tau(a)$. For Lie (G, \mathfrak{g}), $\mathfrak{g} \times \mathfrak{g}^*$ \rightarrow \mathcal{A}^w via $\begin{cases} \text{exp} & \rightarrow \\ \text{ad} & \rightarrow \end{cases}$ \rightarrow $\mathfrak{g}^* \in \text{S}(\mathfrak{g})$. $\text{Fun}(\mathfrak{g}) \xrightarrow{\text{ad}} \text{S}(\mathfrak{g})$
 $\mathfrak{g} \times \mathfrak{g}^*$ \rightarrow \mathcal{A}^w via $\begin{cases} \text{exp} & \rightarrow \\ \text{ad} & \rightarrow \end{cases}$ \rightarrow $\mathfrak{g}^* \in \mathcal{U}(Ig)$. $\text{Fun}(\mathfrak{g}) \xrightarrow{\text{ad}} \mathcal{U}(Ig)$
 $\mathfrak{g} \times \mathfrak{g}^*$ \rightarrow $\mathcal{U}(Ig)$ via $\begin{cases} \text{exp} & \rightarrow \\ \text{ad} & \rightarrow \end{cases}$ \rightarrow $\mathfrak{g}^* \in \mathcal{U}(Ig)$. $\text{Fun}(\mathfrak{g}) \xrightarrow{\text{ad}} \mathcal{U}(Ig)$

Convolution statement (Kashiwara-Vergne): Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately: let G be a finite dimensional Lie group and let \mathfrak{g} be its Lie algebra, let $\varphi : \mathfrak{g} \rightarrow \mathbb{R}$ be the Jacobian of the exponential map $\exp : \mathfrak{g} \rightarrow G$, and let $\Phi : \text{Fun}(\mathfrak{g}) \rightarrow \text{Fun}(G)$ be given by $\Phi(\varphi) = \varphi \circ \exp$. If $f, g \in \text{Fun}(G)$ are Ad-invariant and supported near the identity, then $\Phi(f) * \Phi(g) = \Phi(f * g)$.

We skipped: • The Alexander \star -knots, quantum groups and polynomial Milnor invariants. Also: the proof of the success of the AdS/CFT correspondence, the classification of knot polynomials, and the proof of the success of the DFA and Drinfel'd's associates.

• The simplest problem hyperbolic geometry solves.

Video and more at <http://www.math.toronto.edu/~drorbn/Talks/Bonn-0908/>

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w-Knots from Z to A

Dror Bar-Natan, Lumm, April 2010
<http://www.math.toronto.edu/~drorbn/Talks/Lumm-1004>

Abstract: I will define w-knots, a class of knots wider than ordinary knots but weaker than virtual knots, and show that it is equal to the union of the two main known inclusions of classes of virtual knots. The short: I will introduce the "Euler Operators" and the "Infinite-dimensional Alexander Module", at the end finding a simple determinant formula for Z . With doubt that formula computes the Alexander polynomial, except I don't have a proof yet.

Tubes in 4D: Braiding surface
 Virtual crossing
 Crossing
 Cap Was
 Virtual crossing
 Move
 Vertices
 w-annulus
 w-disk

Corollaries: (1) Related to Lie algebras! (2) Only wheels and isolated arrows persist.
Habiro - can you do better?
The Alexander Theorem: $T_{ij} = \text{low}(\#j) \in \text{span}(\#i)$, $\#i = \text{sign}(\#i)$, $\#d = \text{dir}(\#i)$, $\#(\text{diag}(i)) = 1$, $A = \det(I - T(I - X^{-S}))$, $T = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$, $X^{-S} = \text{tag}(y, X, \frac{\partial}{\partial X}, \frac{\partial}{\partial Y}, \frac{\partial}{\partial Z}, \frac{\partial}{\partial W})$

Conjecture: For u-knots, A is the Alexander polynomial. Theorem: With $w : \mathbb{X}^k \rightarrow \mathbb{X}^k$ ($= \text{the k-wheel}$), $Z = N \exp_{A^w} \left(-w \left(\log_{\text{tag}}(A^w) \right) \right)$ mod $w_{\text{tag}} = w_{\text{tag}} \cdot Z$. $Z = N^{-1} A^{-1} (A^k)$

Proof Sketch: Let E be the Euler operator ("multiply anything by its degree"), $f = xf'$ in $\mathbb{Q}[x]$, $\text{tag}(f) = xf'$ and $\text{tag}(E) = E^k$. We need to show that $Z^* EZ = N^* = \text{tr}(I - B)^{-1} \text{tag}(E^k) w_{\text{tag}}$, with $B = T(e^{-x^2} - I)$. Note that $e^{-x^2} - e^{-y^2} = (1 - e^{-x^2})(1 - e^{-y^2})$ implies $\text{tag}(e^{-x^2} - e^{-y^2}) = \text{tag}(e^{-x^2}) \text{tag}(e^{-y^2})$.

The Finite Type Story: With $\text{V} := Q : \mathbb{V} \otimes \mathbb{V} \rightarrow \mathbb{V}$ set $\mathbb{V}_n := \langle \text{V} : \text{W} \subset Q : \mathbb{V} \otimes \mathbb{V} \rightarrow \mathbb{V} \rangle$.
 arrow diagrams
 $\text{R} = \langle \text{TC}, \text{TC} \rangle \rightarrow \text{D} = \langle \text{TC}, \text{TC} \rangle$
 $\text{A}^w := \text{D}/\text{R}$
 $\text{OC} = \langle \text{TC}, \text{TC} \rangle$
 $\text{R3} = \langle \text{TC}, \text{TC} \rangle$
 $\text{VR3} = \langle \text{TC}, \text{TC} \rangle$
 $\text{D} = \langle \text{TC}, \text{TC} \rangle$
 $\text{OC} = \langle \text{TC}, \text{TC} \rangle$

Theorem: For w-knots, $\dim \mathbb{V}_n/\mathbb{V}_{n-1} = \dim \mathbb{W}_n$ for all n .

Proof: This is the Kontsevich integral, or the "Fundamental Theorem of Finite Type Invariants". The known proofs use QI-inspired differential geometry or associates and some homological computations.

Two tables: The following tables show $\dim \mathbb{V}_n/\mathbb{V}_{n-1}$ and $\dim \mathbb{W}_n$ for $n = 1, \dots, 5$ for 18 classes of w-knots:

mod 1	1	2	3	4	5
mod 2	0	1	2	3	4
mod 3	1	2	3	4	5
mod 4	0	1	2	3	4
mod 5	1	2	3	4	5

Circuit Algebras
 $\text{R1} = \text{C} \otimes \text{Q}$
 $\text{R2} = \text{C} \otimes \text{Q}$
 $\text{R3} = \text{C} \otimes \text{Q}$
 $\text{R4} = \text{C} \otimes \text{Q}$
 $\text{R5} = \text{C} \otimes \text{Q}$
 $\text{R6} = \text{C} \otimes \text{Q}$
 $\text{R7} = \text{C} \otimes \text{Q}$
 $\text{R8} = \text{C} \otimes \text{Q}$
 $\text{R9} = \text{C} \otimes \text{Q}$
 $\text{R10} = \text{C} \otimes \text{Q}$
 $\text{R11} = \text{C} \otimes \text{Q}$
 $\text{R12} = \text{C} \otimes \text{Q}$
 $\text{R13} = \text{C} \otimes \text{Q}$
 $\text{R14} = \text{C} \otimes \text{Q}$
 $\text{R15} = \text{C} \otimes \text{Q}$
 $\text{R16} = \text{C} \otimes \text{Q}$
 $\text{R17} = \text{C} \otimes \text{Q}$
 $\text{R18} = \text{C} \otimes \text{Q}$

Comments: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187, 188, 189, 190, 191, 192, 193, 194, 195, 196, 197, 198, 199, 200, 201, 202, 203, 204, 205, 206, 207, 208, 209, 210, 211, 212, 213, 214, 215, 216, 217, 218, 219, 220, 221, 222, 223, 224, 225, 226, 227, 228, 229, 230, 231, 232, 233, 234, 235, 236, 237, 238, 239, 240, 241, 242, 243, 244, 245, 246, 247, 248, 249, 250, 251, 252, 253, 254, 255, 256, 257, 258, 259, 260, 261, 262, 263, 264, 265, 266, 267, 268, 269, 270, 271, 272, 273, 274, 275, 276, 277, 278, 279, 280, 281, 282, 283, 284, 285, 286, 287, 288, 289, 290, 291, 292, 293, 294, 295, 296, 297, 298, 299, 300, 301, 302, 303, 304, 305, 306, 307, 308, 309, 310, 311, 312, 313, 314, 315, 316, 317, 318, 319, 320, 321, 322, 323, 324, 325, 326, 327, 328, 329, 330, 331, 332, 333, 334, 335, 336, 337, 338, 339, 340, 341, 342, 343, 344, 345, 346, 347, 348, 349, 350, 351, 352, 353, 354, 355, 356, 357, 358, 359, 360, 361, 362, 363, 364, 365, 366, 367, 368, 369, 370, 371, 372, 373, 374, 375, 376, 377, 378, 379, 380, 381, 382, 383, 384, 385, 386, 387, 388, 389, 390, 391, 392, 393, 394, 395, 396, 397, 398, 399, 400, 401, 402, 403, 404, 405, 406, 407, 408, 409, 410, 411, 412, 413, 414, 415, 416, 417, 418, 419, 420, 421, 422, 423, 424, 425, 426, 427, 428, 429, 430, 431, 432, 433, 434, 435, 436, 437, 438, 439, 440, 441, 442, 443, 444, 445, 446, 447, 448, 449, 450, 451, 452, 453, 454, 455, 456, 457, 458, 459, 460, 461, 462, 463, 464, 465, 466, 467, 468, 469, 470, 471, 472, 473, 474, 475, 476, 477, 478, 479, 480, 481, 482, 483, 484, 485, 486, 487, 488, 489, 490, 491, 492, 493, 494, 495, 496, 497, 498, 499, 500, 501, 502, 503, 504, 505, 506, 507, 508, 509, 510, 511, 512, 513, 514, 515, 516, 517, 518, 519, 520, 521, 522, 523, 524, 525, 526, 527, 528, 529, 530, 531, 532, 533, 534, 535, 536, 537, 538, 539, 540, 541, 542, 543, 544, 545, 546, 547, 548, 549, 550, 551, 552, 553, 554, 555, 556, 557, 558, 559, 559, 560, 561, 562, 563, 564, 565, 566, 567, 568, 569, 570, 571, 572, 573, 574, 575, 576, 577, 578, 579, 579, 580, 581, 582, 583, 584, 585, 586, 587, 588, 589, 589, 590, 591, 592, 593, 594, 595, 596, 597, 598, 599, 599, 600, 601, 602, 603, 604, 605, 606, 607, 608, 609, 609, 610, 611, 612, 613, 614, 615, 616, 617, 618, 619, 619, 620, 621, 622, 623, 624, 625, 626, 627, 628, 629, 629, 630, 631, 632, 633, 634, 635, 636, 637, 638, 639, 639, 640, 641, 642, 643, 644, 645, 646, 647, 648, 649, 649, 650, 651, 652, 653, 654, 655, 656, 657, 658, 659, 659, 660, 661, 662, 663, 664, 665, 666, 667, 668, 669, 669, 670, 671, 672, 673, 674, 675, 676, 677, 678, 679, 679, 680, 681, 682, 683, 684, 685, 686, 687, 688, 689, 689, 690, 691, 692, 693, 694, 695, 696, 697, 698, 698, 699, 699, 700, 701, 702, 703, 704, 705, 706, 707, 708, 709, 709, 710, 711, 712, 713, 714, 715, 716, 717, 718, 718, 719, 720, 721, 722, 723, 724, 725, 7