Braverman Lecture 3

July-19-11
9:58 AM

$K(X) = \left\{ \begin{array}{c} E \xrightarrow{\sigma} F \\ \downarrow \\ X \end{array} : \begin{array}{l} \sigma \text{ is invertible outside of a compact} \\ \text{set in } X \end{array} \right\}$

There is a similar $K_G(X)$.

<u>Thm</u> (Atiyah-Singer) $\operatorname{ind}_G \mathcal{D} = t\text{-}\operatorname{ind}_G([\sigma_L(\mathcal{D})]) \in R(G)$.

<u>Q</u> If M is not compact, suppose $[\sigma] \in K_G(T^*M)$

Find $\mathcal{D} : C^\infty(M, E) \to C^\infty(M, F)$ s.t.

$\operatorname{ind}_G(\mathcal{D}) = t\text{-}\operatorname{ind}_G(\sigma)$.

Transversal elliptic operators. Local model:

$N = M \times G \sim (x_1, \ldots, x_n, y_1 \ldots y_m)$

$T^*N \sim (x_1 \ldots x_n, y_1 \ldots y_m, \xi_1 \ldots \xi_n, \eta_1 \ldots \eta_m)$

If $\mathcal{D}$ is elliptic on M, $\tilde{\mathcal{D}}$ its extension to N, then

$$\ker \tilde{\mathcal{D}} = \ker \mathcal{D} \otimes L^2(G)$$

$$L^2(G) = \bigoplus_{V_i \in \operatorname{Irr}(G)} l_i V_i \qquad l_i = \dim V_i$$

So $\ker \tilde{\mathcal{D}} = \bigoplus m_i^+ V_i \in \hat{R}(G)$

$\qquad\qquad\qquad\qquad \uparrow$
$\qquad\qquad\qquad\qquad \mathbb{Z}^+$

If $G \circlearrowright M$, set $T^*_G(M) = \{\xi \in T^*M : \xi \perp \text{orbits}\}$

... There is a notion of "transversal ellipticity"

Thm In that case,
(Atiyah) 70~71

$$\ker D = \widehat{\bigoplus} m_i^+ V_i \quad \operatorname{coker} D = \widehat{\bigoplus} m_i^- V_i \, ,$$

so $\operatorname{ind}_G D \in \hat{R}(G)$ makes sense (and remains stable).

... there is also a $t\text{-}\operatorname{ind}_G : K_G^*(T_G^* M) \to \hat{R}(G)$, and an Atiyah-Singer index theorem.

Now $G \curvearrowright M$, M non-compact. Suppose also we have an equivariant

$$v : M \to \mathfrak{g} \cong \mathfrak{g}^*$$

$\Rightarrow$ this defines an equivariant vector field V_M on M.

Assumption: $V_M \neq 0$ outside of a compact set.