We shall need a Hopf algebra version of the above definition. In this case \( A_0 \) is a Poisson-Hopf algebra (i.e., a Hopf algebra structure and a Poisson algebra structure on \( A_0 \) are given such that the multiplication is the same for both structures and the comultiplication \( A_0 \to A_0 \otimes A_0 \) is a Poisson algebra homomorphism, the Poisson bracket on \( A_0 \otimes A_0 \) being defined by \( \langle a \otimes b, c \otimes d \rangle = ac \otimes \{ b, d \} + \{ a, c \} \otimes bd \) and \( A \) is a Hopf algebra deformation of \( A_0 \). We shall also use the dual notion of quantization of co-Poisson-Hopf algebras (a co-Poisson-Hopf algebra is a cocommutative Hopf algebra \( B \) with a Poisson cobracket \( B \to B \otimes B \) compatible with the Hopf algebra structure).

We discuss the structure of Poisson-Hopf algebras and co-Poisson-Hopf algebras in §§3 and 4. Then we consider the quantization problem.

3. Poisson groups and Lie bialgebras. A Poisson group is a group \( G \) with a Poisson bracket on \( \text{Fun}(G) \) which makes \( \text{Fun}(G) \) a Poisson-Hopf algebra. In other words the Poisson bracket must be compatible with the group operation, which means that the mapping \( \mu : G \times G \to G \), \( \mu(g_1, g_2) = g_1 g_2 \), must be a Poisson mapping in the sense of [33], i.e., \( \mu^* : \text{Fun}(G) \to \text{Fun}(G \times G) \) must be a Lie algebra homomorphism. Specifying the meaning of the word “group” and the symbol \( \text{Fun}(G) \), we obtain the notions of Poisson-Lie group, Poisson formal group, Poisson algebraic group, etc. According to our general principles the notions of Poisson group and Poisson-Hopf algebra are equivalent.

There exists a very simple description of Poisson-Lie groups in terms of Lie bialgebras.

**Definition.** A Lie bialgebra is a vector space \( g \) with a Lie algebra structure and a Lie coalgebra structure, these structures being compatible in the following sense: the cocommutator mapping \( g \to g \otimes g \) must be a 1-cocycle (\( g \) acts on \( g \otimes g \) by means of the adjoint representation).

If \( G \) is a Poisson-Lie group then \( g = \text{Lie}(G) \) has a Lie bialgebra structure. To define it write the Poisson bracket on \( C^\infty(G) \) as

\[
\{ \varphi, \psi \} = \eta^{\mu \nu} \partial_\mu \varphi \cdot \partial_\nu \psi, \quad \varphi, \psi \in C^\infty(G),
\]

where \( \{ \partial_\mu \} \) is a basis of right-invariant vector fields on \( G \). The compatibility of the bracket with the group operation means that the function \( \eta : G \to g \otimes g \) corresponding to \( \eta^{\mu \nu} \) is a 1-cocycle. The 1-cocycle \( f : g \to g \otimes g \) corresponding to \( \eta \) defines a Lie bialgebra structure on \( g \) (the Jacobi identity for \( f^* : g^* \otimes g^* \to g^* \) holds because \( f^* \) is the infinitesimal part of the bracket (4)).

**Theorem 1.** The category of connected and simply-connected Poisson-Lie groups is equivalent to the category of finite dimensional Lie bialgebras.

The analogue of Theorem 1 for Poisson formal groups over a field of characteristic 0 can be proved in the following way. The algebra of functions on the formal group corresponding to \( g \) is nothing but \( (Ug)^* \). A Poisson-Hopf structure on \( (Ug)^* \) is equivalent to a co-Poisson-Hopf structure on \( Ug \). So it suffices to prove the following easy theorem.

**Theorem 2.** Let \( \delta : Ug \to Ug \otimes Ug \) be a Poisson cobracket which makes \( Ug \) a co-Poisson-Hopf algebra (the Hopf structure on \( Ug \) is usual). Then \( \delta(g) \subset g \otimes g \) and \( g, \delta(g) \) is a Lie bialgebra. Thus we obtain a one-to-one correspondence between co-Poisson-Hopf structures on \( Ug \) inducing the usual Hopf structure and Lie bialgebra structures on \( g \) inducing the given Lie algebra structure.

**Question.** Is there a diagrammatic version of that?