$\mathbb{F}_p[[t]]$ – a topological ring (w/ product topology) has a canonical probability measure λ.

Question $\lambda(\{y^2 : y \in \mathbb{F}_p[[t]]\}) = ?$

Valuation $Val(x) = k$ if $x = \sum_{j \geq k} a_j t^j$ $a_k \neq 0$

$|x| = p^{-k}$

$ac(x) = a_k$

"angular components"

properties: $|x||y| = |xy|$, $ac(xy) = ac(x)ac(y)$

So x is a square $\Rightarrow$ $val(x)$ is even, $ac(x)$ is a square.

Claim If $p > 2$, this is $\Leftrightarrow$.

Anyway, the above measure is (if $p>2$) $\sum_{k=0}^{\infty} \sum_{\alpha \in \mathbb{F}_p^{*2}} \frac{1}{p^{2k+1}}$

$$= \underbrace{|\mathbb{F}_p^{*2}|}_{\text{depends on a finite thing}} \cdot \underbrace{\sum_{k=0}^{\infty} \frac{1}{p^{2k+1}}}_{\text{Infinite geometric sum.}}$$

Generalization.

$\mathcal{L}$ = The first order language using $+, \cdot, val, ac, \leq$ (where $\leq$ is used only for integers)

The language has objects of type int, $\mathbb{F}_p$, $\mathbb{F}_p[[t]]$

"multi-sorted logic".

Def A "simple" formula $\phi(x_1, \dots, x_n)$ $(x_i \in \mathbb{F}_p[[t]])$ is a formula of the form

$$\varphi(ac(x_1), \dots, ac(x_n)) \wedge \psi(val(x_1), \dots, val(x_n))$$

In general, write

In general, write

"definable set" $\begin{cases} X_\phi(\mathbb{F}_p[[t]]) = [\phi(x_1,\ldots,x_n)] = \\ \qquad = \{(x_1,\ldots,x_n) : \phi = \text{true}\} \end{cases}$

Main Thm. Every definable set X has a decomposition $X = X_1 \amalg \ldots \amalg X_N$ s.t. every X_i is in measure preserving bijection w/ a <u>simple</u> definable set. (speaker does not know if this is effective) (if p is large enough)

$$\lambda(\{\vec{x} \mid \eta(ac(\vec{x})) \wedge \xi(val(\vec{x}))\}) =$$

$$\sum_{\substack{(d_1\ldots d_n)\in\mathbb{Z}^n \\ \text{sat } \xi}} \cdot \sum_{\substack{(\alpha_1\ldots\alpha_n)\in(\mathbb{F}_p^*)^n \\ \text{sat } \eta}} p^{-\sum d_i+1} = |\{\vec{\alpha} \mid \eta(\vec{\alpha})\}| \cdot \begin{pmatrix} \text{w.l.o.g.} \\ \text{a sum} \\ \text{over a} \\ \text{cone} \end{pmatrix}$$

$\Rightarrow$ Volumes of definable sets are always rational.

Exciting Thing. The decomposition in **main** is uniform!

$\mathbb{F}_p[[t]] = \varprojlim \mathbb{F}_p[[t]]/(t^n)$ If p is large enough, the same decomposition works also for $\mathbb{Z}_p$.

$\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n$

literally the same, using the same formulas.

<u>Corollary</u> If X is definable then (for large p)

$$\lambda(X(\mathbb{F}_p[[t]])) = \lambda(X(\mathbb{Z}_p))$$

in particular, every first order statement true in one is true in the other.

Why "motivic"?

$$\lambda(X(\mathbb{F}_p[[t]])) = \sum_{i=1}^{N} |Y_i(\mathbb{F}_p)| \Big(\sum_{\substack{\text{cone} \\ C_i}} \frac{1}{p^{\ell_i}} \Big)$$

$\rightsquigarrow$ " $\lambda(X) = \sum [Y_i][c_i, \varphi_i]$

true is some formal quotient like $[Y_1 \cup Y_2] = [Y_1] + [Y_2]$