Halacheva: Link Invariants from Ribbon Categories

September-29-10
9:17 AM

Recall $(\mathcal{C}, \otimes, 1, \alpha, \sigma)$

Plan: Rigid monoidal categories $\longrightarrow$ Ribbon category

"dual V^*" $\qquad\qquad\qquad V \cong V^{**}$

$\longrightarrow$ graphical calculus $\longrightarrow$ Reshetikhin-Turaev Invariants

(present morphisms as framed tangles)

<u>Def</u> $\mathcal{C}$ - monoidal category, $V \in Ob(\mathcal{C})$. A right dual to V is $V^* \in Ob(\mathcal{C})$ with

$$e_V : V^* \otimes V \longrightarrow 1 \qquad i_V : 1 \longrightarrow V \otimes V^*$$

s.t.

$$\begin{tikzcd} & V \otimes V^* \otimes V \arrow[d, "I \otimes e_V"] \\ V \arrow[ur, "i_V \otimes I"] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & V \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & V^* \otimes V \otimes V^* \arrow[dr] & \\ V^* \arrow[ur] \arrow[rr] \arrow[rr, phantom, "\circlearrowleft"] & & V^* \end{tikzcd}$$

<u>Example</u> 1. $Vect_F$: f.d. v.s.

<u>Def</u> Left dual : $e_V' : V \otimes {}^*V \longrightarrow 1$, $i_V' : 1 \longrightarrow {}^*V \otimes V$

<u>Def</u> A rigid monoidal category is a monoidal category in which every object has a right and a left dual. ($Vect_F$ is still an example)

<u>Lemma</u> $\mathrm{Hom}(U \otimes V, W) \cong \mathrm{Hom}(U, W \otimes V^*)$

$\mathrm{Hom}(U, V \otimes W) \cong \mathrm{Hom}(V^* \otimes U, W)$

$\mathrm{Hom}(U, V) \cong \mathrm{Hom}(V^*, U^*)$

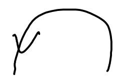
<u>Def</u> A ribbon category is a rigid braided monoidal category with natural isomorphism $\delta : V \to V^{**}$ s.t.

1. $\delta_{V \otimes W} = \delta_V \otimes \delta_W$
2. $\delta_1 = Id$ $\qquad\qquad$ (E.g. $Vect_F$)

3. $\delta_{V^*} = (\delta_V^*)^{-1}$

III Graphical Calculus

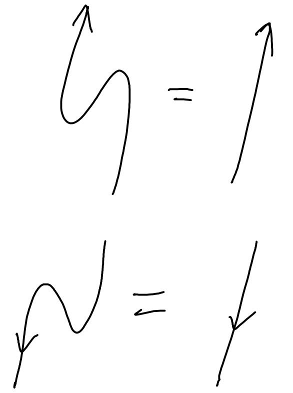
$e_V : V^* \otimes V \to 1$



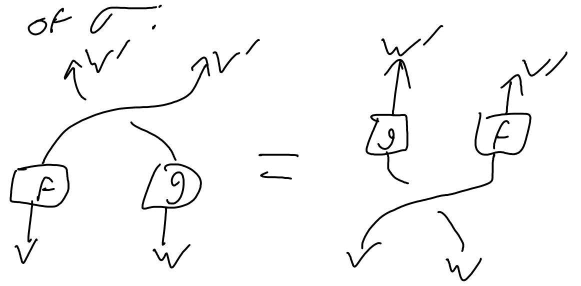
$i_V : 1 \to V \otimes V^*$

$\sigma_{VW} : V \otimes W \to W \otimes V$

The relations:



The functoriallity of σ:



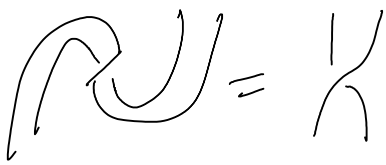
where $f : V \to V'$, $g : W \to W'$

Lemma

$$\begin{tikzcd} V \otimes W^* \otimes W \arrow[r, "\sigma_{V, W^* \otimes W}"] \arrow[d] & W^* \otimes W \otimes V \arrow[d] \\ V \otimes 1 \arrow[r, "\sigma_{V,1}"] & 1 \otimes V \end{tikzcd}$$

$\rightarrow$

Lemma $(\sigma_{VW})^* = \sigma_{V^*, W^*}$



Def $V \in \mathrm{Obj}(\mathcal{C})$, $f \in \mathrm{End}(V)$, $tr(f) := e_{V^*} \circ (\delta \otimes 1) \circ (f \otimes I) \circ i_V$
$(\in \mathrm{End}(1) = \mathbb{F})$

Def $\dim V := tr(Id_V)$.

Claim Isotopic framed $\mathcal{C}$-coloured tangles $\rightarrow$ equal morphisms.

Aside If $\mathcal{C}$ is symmetric,