Chicago ax+b Talk as of September 2

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From the $ax+b$ Lie Algebra to the Alexander Polynomial and Beyond

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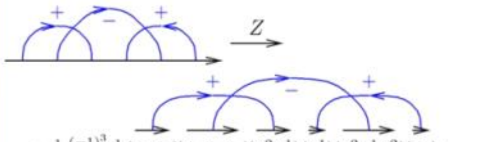
http://www.math.toronto.edu/~drorbn/Talks/Chicago-1009/

Abstract. I will present the simplest-ever "quantum" formula for the Alexander polynomial, using only the unique two dimensional non-commutative Lie algebra (the one associated with the "$ax+b$" Lie group). After introducing some diagrammatic calculus I will sketch the proof of the said formula, and following that, I will present a long list of extensions, generalizations, and dreams.

The 2D Lie Algebra. Let $\mathfrak{g} = \mathfrak{lie}(x^1, x^2)/[x^1, x^2] = x^2$, let $\mathfrak{g}^* = \langle \phi_1, \phi_2 \rangle$ with $\phi_i(x^j) = \delta_i^j$, let $I\mathfrak{g} = \mathfrak{g}^* \rtimes \mathfrak{g}$ so $[\phi_i, \phi_j] = [\phi_1, x^i] = 0$ while $[x^1, \phi_2] = -\phi_2$ and $[x^2, \phi_2] = \phi_1$. Let $r = Id = \phi_1 \otimes x^1 + \phi_2 \otimes x^2 \in \mathfrak{g}^* \otimes \mathfrak{g} \subset I\mathfrak{g} \otimes I\mathfrak{g}$. Let $\mathcal{U} = \{\text{words in } I\mathfrak{g}\}/ab - ba = [a,b]$, degree-completed with respect to $\deg \phi_i = 1$ and $\deg x^i = 0$ (so $\mathcal{U} \equiv$ (power series is 4 variables)). Let $R = \exp(r) \in \mathcal{U} \otimes \mathcal{U}$.

The Invariant. Define $Z : \{\text{long knots}\} \to \mathcal{U}$ by mapping every $\pm$-crossing to $R^{\pm 1}$:

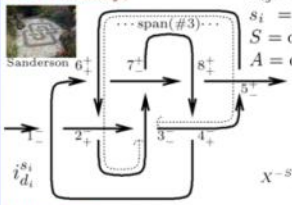


$$\cdots + \tfrac{1}{2!}\tfrac{(-1)^3}{3!}\tfrac{1}{1!}(\phi_2\phi_1)(\phi_2\phi_1\phi_2)(x^2x^1)(x^1)(x^2x^1x^2)(\phi_1) + \cdots$$

The Theorem. Z is invariant, and it is essentially the Alexander polynomial; with $N = \exp(\overrightarrow{l}\phi_i x^i + \overleftarrow{l} x^i \phi_i) =: \exp(SL)$,

$$\tilde{Z}(K) = N \cdot \left(A(K)(e^{\phi_1})\right)^{-1}. \qquad (1)$$

More Honestly,



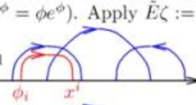
$T_{ij} = |\text{low}(\#j) \in \text{span}(\#i)|$,
$s_i = \text{sign}(\#i)$, $d_i = \text{dir}(\#i)$,
$S = \text{diag}(s_i d_i)$, $B = T(X^{-S} - I)$
$A = \det(I - B)$.

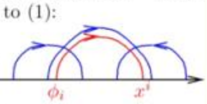
$$T = \begin{pmatrix} 0&1&1&1&1&0&1&0 \\ 0&0&1&0&1&0&0&0 \\ 0&1&0&0&1&0&0&0 \\ 0&1&0&0&1&0&1&0 \\ 0&1&0&1&0&1&1&1 \\ 0&1&0&1&0&0&1&0 \\ 0&0&0&1&0&1&0&0 \\ 0&0&0&1&0&1&0&0 \end{pmatrix},$$

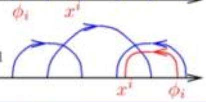
$$X^{-S} = \text{diag}(\tfrac{1}{X}, X, \tfrac{1}{X}, X, X, \tfrac{1}{X}, X, \tfrac{1}{X}).$$

Conjecture. For u-knots, A is the Alexander polynomial.

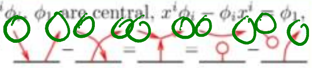
An Euler Prelude. Let E be the Euler derivation, which multiplies each element by its degree (e.g. on $\mathbb{Q}[\![\phi]\!]$, $Ef = \phi\partial_\phi f$, so $Ee^\phi = \phi e^\phi$). Apply $\tilde{E}\zeta := \zeta^{-1}E\zeta$ to (1):

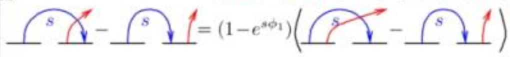
$+Z^{-1}$ 

$-Z^{-1}$ 

$+Z^{-1}$ 

$\stackrel{?}{=} SL$
$-\phi_1 \operatorname{tr}(1-B)^{-1}TSe^{-\phi_1 S}$
at $X = e^{\phi_1}$

Some Relations. $\phi_i x^i$, $x^i\phi_i$, ϕ_1 are central, $x^i\phi_i - \phi_i x^i = \phi_1$, $[x^j, \phi_i] = \delta_i^j\phi_1 - \delta_1^j\phi_i$ or 
so



and the hallmark "tails commute": 

add indices.

"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

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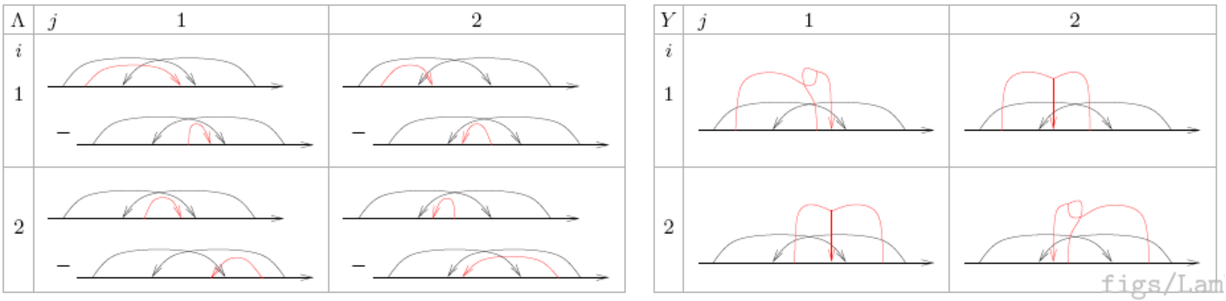

Figure 18. The matrices Λ and Y for a sample 2-arrow Gauss diagram (the signs on a_1 and a_2 are suppressed, and so are the r marks). The twists in y_{11} and y_{22} may be replaced by minus signs.

From papers/wko

entries in IAM_G hold true:

$$\begin{aligned} \lambda - SL &= \operatorname{tr} S\Lambda & (25)\\ \Lambda &= -BY - TX^{-S}w_1 & (26)\\ Y &= BY + TX^{-S}w_1 & (27) \end{aligned}$$

$$\Lambda = e^{ind \cdot \phi} SL + T(I - e^{ind})\Lambda + T\phi$$

$$\lambda = tr \Lambda$$

$$\Lambda = (I - T(I - e^{ind}))^{-1} (T\phi + e^{ind \phi} SL)$$