Some A-T Notions: $a_n$ is the vector space with basis $x_1, \ldots, x_n$. Lie $a_n = \text{Lie}(a_n)$ is the free Lie algebra, $\text{Ass}_n = \text{Lie}(\text{tr}_n)$ is the free associative algebra “of words”, $\text{tr} : \text{Ass}_n \rightarrow \text{tr}_n = \text{Ass}_n^\circ / \langle x_1 x_2 \ldots x_i x_j x_{i+1} \ldots x_l x_{i+2} \rangle$ is the “trace” into the cyclic words, $\text{ter}_n = \text{ter}(\text{tr}_n)$ are all the derivations, and $\partial \text{er}_n = \{ D \in \text{ter}_n \mid \exists \alpha \text{ s.t. } D(x_i) = [x_i, \alpha] \}$ are “tangential derivations”, so $D \rightarrow (a_1, \ldots, a_n)$ is a vector space isomorphism $a_n \oplus \partial \text{er}_n \cong \bigoplus_{n} \text{tr}_n$. Finally, $\partial \text{er}_n \rightarrow \text{tr}_n$ is $\sum \text{tr}(x_i 0(a_n))$, where for $a \in \text{Ass}_n^\circ$ $\partial \text{er}_n a \in \text{Ass}_n$ is determined by $a = \sum_i (0(a_i) x_i)$ and $j : \text{Taut} \rightarrow \text{tr}_n \rightarrow \text{tr}_n$ is $j(e^D) = e^{\text{tr}D}$. div $D = 0$.

Theorem. Every matching (trees) is $a_n \oplus \partial \text{er}_n$, as Lie algebras, (wheels) is $\text{tr}_n$, as (trees) / $(\partial \text{er}_n)$-modules, $\text{div} D = e^{\text{tr}D}(1 - t)(D)$, and $e^{\text{tr}D} e^{\text{tr}D} = e^{\text{tr}D}$.

Differential Operators. Interpret $\mathcal{U}(g)$ as tangential differential operators on $\text{Fun}(g)$:

- $\varphi \in g^*$ becomes a multiplication operator.
- $x \in g$ becomes a tangential derivation, in the direction of the action of ad $x : (x \varphi(y)) := [x, y] \varphi(y)$.

Trees become vector fields and $u D \rightarrow (D)$ is $D \rightarrow D^\ast$. So div $D = -D^\ast$ and $jD = \log(e^{\text{tr}D}) = \int e^{tD} \text{div} D$.

Special Derivatives. Let $\text{er}_n = \{ D \in \text{ter}_n : D(x_1) = 0 \}$.

Theorem. $\text{er}_n = \text{tr}(\text{proj} n\text{-tangles})$, where $\alpha$ is the obvious map $\text{proj} n\text{-tangles} \rightarrow \text{proj} n\text{-tangles}$.

Proof. After decoding, this becomes Lemma 6.1 of Drinfeld’s amazing Gal(Q/Q) paper.

The Alexander Theorem.

Conjecture. For $n$-knots, $A$ is the Alexander polynomial.

This is the ultimate Alexander invariant! Computable in polynomial time, local, composes well, behaves under cabling. Seems to significantly generalize the multivariable Alexander polynomial and the theory of Milnor linking numbers. But it’s ugly, and much work remains.
* See that A-B pair appear across folds.