Definition 6.5. A quandle is a set $Q$ with a binary operation $\uparrow : Q \times Q \to Q$ satisfying the following axioms:

1. $\forall x \in Q, x \uparrow x = x$.
2. For any fixed $y \in Q$, the map $x \mapsto x \uparrow y$ is invertible.
3. Self-distributivity: $\forall x, y, z \in Q, (x \uparrow y) \uparrow z = (x \uparrow z) \uparrow (y \uparrow z)$.

We say that a quandle $Q$ has a unit, or is unital, if there is a distinguished element $1 \in Q$ satisfying the further axiom:

4. $\forall x \in Q, x \uparrow 1 = x$ and $1 \uparrow x = 1$.

If $G$ is a group, it is also a (unital) quandle by setting $x \uparrow y := y^{-1}xy$, yet there are many quandles that do not arise from groups in this way.

Proposition 6.6. If $Q$ is a unital quandle, $\text{proj}_0 Q$ is one-dimensional and $\text{proj}_{\uparrow} Q$ is a graded Lie algebra generated by $\text{proj}_1 Q$.

Proof. For any algebraic structure $A$ with just one kind of objects, $\text{proj}_0 A$ is one-dimensional, generated by the equivalence class $[x]$ of any single object $x$. In particular, $\text{proj}_0 Q$ is one-dimensional and generated by $[1]$. Let $I \subseteq \mathbb{Q} Q$ be the augmentation ideal of $Q$. For any $x \in Q$ set $\mathcal{I} := x - 1 \in \mathcal{I}$. Then $\mathcal{I}$ is generated by the $x$’s, and therefore $\mathcal{I}^m$ is generated by expressions involving the operation $\uparrow$ applied to some $m$ elements of $Q := \{ x : x \in Q \}$ and possibly some further elements $y_i \in Q$. When regarded in $\mathcal{I}/\mathcal{I}^m$, any $y_i$ is a generating expression can be replaced by 1, for the difference would be the same expression with $y_i$ replaced by $y_i$, and this is now a member of $\mathcal{I}^m$. But for any element $z \in \mathcal{I}$ we have $z \uparrow 1 = z$ and $1 \uparrow z = 0$, so all the $1$’s can be eliminated from the expressions generating $\mathcal{I}^m$. Thus $\text{proj}_{\uparrow} Q$ is generated by $\mathcal{I}$ and hence by $\text{proj}_1 Q$.

Let $\Delta : \mathbb{Q} Q \to \mathbb{Q} Q \otimes \mathbb{Q} Q$ be the linear extension of the operation $x \mapsto x \otimes x$ defined on $x \in Q$, and extend $\uparrow$ to a binary operator $\uparrow_2 : (\mathbb{Q} Q \otimes \mathbb{Q} Q) \otimes (\mathbb{Q} Q \otimes \mathbb{Q} Q) \to \mathbb{Q} Q \otimes \mathbb{Q} Q$ by using $\uparrow$ twice, to pair the first and third tensor factors and then to pair the second and the fourth tensor factors. With this language in place, the self-distributivity axiom becomes the following linear statement, which holds for every $x, y, z \in \mathbb{Q} Q$:

$$ (x \uparrow y) \uparrow z = \uparrow_2 (x \otimes y \otimes \Delta z). \quad (25) $$

Clearly, we need to understand $\Delta$ better. By direct computation, if $x \in Q$ then $\Delta x = x \otimes 1 + 1 \otimes x + x \otimes x$. We claim that in general, if $z$ is a generating expression of $\mathcal{I}^m$ (that is, a formula made of $m$ elements of $Q$ and $m - 1$ applications of $\uparrow$), then

$$ \Delta z = z \otimes 1 + 1 \otimes z + \sum z_i \otimes z_i', \quad \text{with} \quad \sum z_i \otimes z_i' \in \bigoplus_{m, m'} \mathcal{I}^{m} \otimes \mathcal{I}^{m'}. \quad (26) $$

Indeed, for the generators of $\mathcal{I}$ this had just been shown, and if $z = z_1 \uparrow z_2$ is a generator of $\mathcal{I}^m$, with $z_1$ and $z_2$ generators of $\mathcal{I}^{m_1}$ and $\mathcal{I}^{m_2}$ with $1 \leq m_1, m_2 < m$ and $m_1 + m_2 = m$,

---

\footnote{This can alternatively be stated as “there exists a second binary operation $\uparrow^{-1}$ so that $\forall x, y = (x \uparrow y)^{-1}$, so this axiom can still be phrased within the language of “algebraic structures”. Yet note that below we do not use this axiom at all.}

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then (using \( w \uparrow 1 = w \) and \( 1 \uparrow w = 0 \) for \( w \in \mathcal{I} \)),

\[
\Delta z = \Delta(z_1 \uparrow z_2) = (\Delta z_1) \uparrow (\Delta z_2)
\]

\[
= (z_1 \uparrow 1 \uparrow z_1 + \sum \delta_{ij} \uparrow \hat{z}_{ij} \uparrow z_{ij}) \uparrow (z_2 \uparrow 1 \uparrow z_2 + \sum \delta_{ik} \uparrow \hat{z}_{ik} \uparrow z_{ik})
\]

\[
= (z_1 \uparrow z_2) \uparrow 1 \uparrow (z_1 \uparrow z_2) + \sum_j \left( (z_1 \uparrow z_2) \uparrow z_{ij} + z_{ij} \uparrow (z_1 \uparrow z_2) + \sum_k (\delta_{ij} \uparrow \hat{z}_{ik}) \uparrow (\delta_{ij} \uparrow \hat{z}_{ik}) \right)
\]

and it is easy to see that the last line agrees with (26).

We can now combine Equations (25) and (26) to get that for any \( x, y, z \in \mathbb{Q} \),

\[
(x \uparrow y) \uparrow z = (x \uparrow z) \uparrow y + x \uparrow (y \uparrow z) + \sum (x \uparrow z) \uparrow (y \uparrow z).
\]

If \( x \in \mathcal{I}^{m_1} \), \( y \in \mathcal{I}^{m_2} \), and \( z \in \mathcal{I}^{m_3} \), then by (26) the last term above is in \( \mathcal{I}^{m_1 + m_2 + m_3 + 1} \), and so the above identity becomes the Jacobi identity \((x \uparrow y) \uparrow z = (x \uparrow z) \uparrow y + x \uparrow (y \uparrow z) \) in \( \text{proj}_{m_1 + m_2 + m_3 + 1} \mathcal{I} \).

MORE. It remains to show that within \( \text{proj}_{m_1} \mathcal{I} \), the operation \( \uparrow \) is anti-symmetric.

Exercise 6.7. Verify that in the above proof axiom (2) of Definition 6.5 was not used. Verify also that if this axiom is introduced as in footnote 29 using a second operation \( \uparrow^{-1} \) (thus enlarging the set of algebraic expressions that we need to consider as in MORE), Proposition 6.6 remains true.

\[
\text{Compare with Leibnitz:}
\]

\[
[x \uparrow y \uparrow z] = [x \uparrow y] \uparrow z - [x \uparrow z] \uparrow y \rightarrow \text{Flip} \ [,] \]

\[
[x \uparrow y] \uparrow x = [z \uparrow y \uparrow x] - [y \uparrow z] \uparrow x \rightarrow (x \uparrow z) \uparrow (y \uparrow z)
\]

\[
[x \uparrow y] \uparrow z = [x \uparrow y] \uparrow z - [y \uparrow z] \uparrow x \rightarrow \text{It would be lovely to know that}
\]

\[
(x \uparrow z) \uparrow y = -y \uparrow (x \uparrow z)
\]