The Core E-K Argument, Following Haviv

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5.1.5. **The maps $i_\pm : \mathcal{M}_\pm \to \mathcal{M}_\pm \boxtimes \mathcal{M}_\pm$.** Two more $\vec{\mathcal{A}}$-maps we will use below are the maps $i_\pm : \mathcal{M}_\pm \to \mathcal{M}_\pm \boxtimes \mathcal{M}_\pm$. The map i_+ is defined as follows:

$$i_+(p_+(D)) = (p_+ \boxtimes p_+)(\Delta(D)), \quad D \in \vec{\mathcal{A}}\,.$$

It is immediate to check that i_+ is well-defined and is an $\vec{\mathcal{A}}$-map, namely

$$i_+(DE) = \Delta(D) i_+(E)\,,$$

for all $D \in \vec{\mathcal{A}}, E \in \mathcal{M}_+$. Similar definition and properties are shared by the map i_-.

5.2. **The diagrammatic Etingof-Kazhdan twist.** Following [EK1], we construct an element $J \in \vec{\mathcal{A}}^{\boxtimes 2}$ as follows. Put

$$\widetilde{J} = \mathbf{Z}_{\mathrm{K}}\Big(\ \ \Big) \in \mathcal{A}^{\boxtimes 4}\,,$$

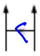
and use the same symbol to denote $\iota(\widetilde{J}) \in \vec{\mathcal{A}}^{\boxtimes 4}$. Set

$$J = (\phi^{-1} \boxtimes \phi^{-1})(p_+ \boxtimes p_- \boxtimes p_+ \boxtimes p_-)(\widetilde{J}) \in \vec{\mathcal{A}}^{\boxtimes 2}\,.$$

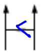
By expressing the parenthesized braid in the formula for $\widetilde{J}$ as the composition of generating associativity and braiding morphisms, and recalling (4.33), we see that

$$\widetilde{J} = \uparrow\ \uparrow\ \uparrow\ \uparrow\ + \tfrac{1}{2}\ \ \ + \text{ terms of degree } > 1\,,$$

and it is then easy to check that

$J =$  $+ \tfrac{1}{2}$  $+$ terms of degree $> 1\,.$ (5.7)

Theorem 7. *Twisting $\vec{\mathcal{A}}_{KZ}$ by $F = J^{-1}$ yields a quasitriangular* Hopf *structure on $\vec{\mathcal{A}}$, denoted $\vec{\mathcal{A}}_{EK}$, whose R-matrix is a quantization of the diagrammatic r-matrix:*

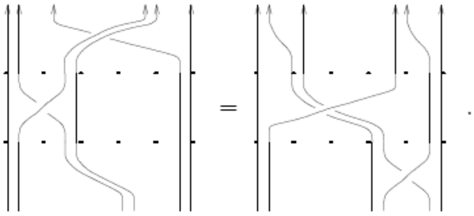
$R_{EK} =$  $+$  $+$ *terms of degree* $> 1\,.$ (5.8)

Proof. By the transformation formula (4.26) of the associator under twisting, we have to show that

$$\Phi\cdot(\Delta\boxtimes\mathrm{id})(J)\cdot J^{12} = (\mathrm{id}\boxtimes\Delta)(J)\cdot J^{23}\,, \tag{5.9}$$

in order to prove that $\vec{\mathcal{A}}_{\mathrm{EK}}$ is Hopf (that is, its coproduct is coassociative).

Our starting point is the topological equivalence of the following two parenthesized braids:



The order of the terms is opposite between these two.

Applying $\mathbf{Z}_{\mathrm{K}}$ to both sides, using Definition 4.35 and the definition of $\tilde{J}$, we find that

$$\Delta^{\boxtimes 3}(\Phi)\cdot(\tilde{J}\boxtimes 1^{\boxtimes 2})\cdot((\Delta^{\boxtimes 2}\boxtimes\mathrm{id}^{\boxtimes 2})(\tilde{J}))^{(23)} = (1^{\boxtimes 2}\boxtimes\tilde{J})\cdot((\mathrm{id}^{\boxtimes 2}\boxtimes\Delta^{\boxtimes 2})(\tilde{J}))^{(45)}\cdot(\Phi^{135}\boxtimes\Phi^{246})$$

in $\mathcal{A}^{\boxtimes 6}$. Here we have used the following notation: for an element X in some (diagrammatic) tensor product, $X^{(ij)}$ is the element obtained from X by transposing its ith and jth factors.

We map the last equality into $\vec{\mathcal{A}}^{\boxtimes 6}$ (using $\iota : \mathcal{A}^{\boxtimes 6}\to\vec{\mathcal{A}}^{\boxtimes 6}$) and then apply the map $(p_+\boxtimes p_-)^{\boxtimes 3} : \vec{\mathcal{A}}^{\boxtimes 6}\to(\mathcal{M}_+\boxtimes\mathcal{M}_-)^{\boxtimes 3}$. Since $p_\pm$ are $\vec{\mathcal{A}}$-maps we get:

$$\begin{aligned}\Delta^{\boxtimes 3}(\Phi)\cdot(p_+\boxtimes p_-)^{\boxtimes 3}((\tilde{J}\boxtimes 1^{\boxtimes 2})\cdot((\Delta^{\boxtimes 2}\boxtimes\mathrm{id}^{\boxtimes 2})(\tilde{J}))^{(23)}) =&\\ (1^{\boxtimes 2}\boxtimes\tilde{J})\cdot((\mathrm{id}^{\boxtimes 2}\boxtimes\Delta^{\boxtimes 2})(\tilde{J}))^{(45)}\cdot(p_+\boxtimes p_-)^{\boxtimes 3}(\Phi^{135}\boxtimes\Phi^{246})\,.&\end{aligned}$$

The crucial observation now (compare [EK1, Lemma 2.3]) is that

$$(p_+\boxtimes p_-)^{\boxtimes 3}(\Phi^{135}\boxtimes\Phi^{246}) = (1_+\boxtimes 1_-)^{\boxtimes 3}\,.$$

This follows from the fact that Φ is horizontal and the definitions of ι and $p_\pm$. We thus conclude that the following equality holds in $(\mathcal{M}_+\boxtimes\mathcal{M}_-)^{\boxtimes 3}$:

$$\begin{aligned}\Delta^{\boxtimes 3}(\Phi)\cdot(p_+\boxtimes p_-)^{\boxtimes 3}((\tilde{J}\boxtimes 1^{\boxtimes 2})\cdot((\Delta^{\boxtimes 2}\boxtimes\mathrm{id}^{\boxtimes 2})(\tilde{J}))^{(23)}) =&\\ (p_+\boxtimes p_-)^{\boxtimes 3}((1^{\boxtimes 2}\boxtimes\tilde{J})\cdot((\mathrm{id}^{\boxtimes 2}\boxtimes\Delta^{\boxtimes 2})(\tilde{J}))^{(45)})\,.&\end{aligned} \tag{5.10}$$

We introduce an auxiliary map $\psi : \mathcal{M}_+\boxtimes\mathcal{M}_-\to(\mathcal{M}_+\boxtimes\mathcal{M}_-)^{\boxtimes 2}$, which is defined by

$$\psi(E) = (\tilde{J}^{(23)}\cdot(i_+\boxtimes i_-)(E))^{(23)}\,,\quad E\in\mathcal{M}_+\boxtimes\mathcal{M}_-\,.$$

might ψ be "the missing principle"?

Note that

$$\psi(1_+\boxtimes 1_-) = \tilde{J}\cdot(1_+\boxtimes 1_-)^{\boxtimes 2} = \phi^{\boxtimes 2}(J)\,. \tag{5.11}$$

We claim that ψ is an $\vec{\mathcal{A}}$-map: Since $i_\pm$ are $\vec{\mathcal{A}}$-maps,

$$\psi(\Delta(D)\cdot E) = (\tilde{J}^{(23)}\cdot \Delta^{(4)}(D)\cdot (i_+ \boxtimes i_-)(E))^{(23)}\,.$$

for $D\in\vec{\mathcal{A}}, E\in \mathcal{M}_+\boxtimes\mathcal{M}_-$. Since $\tilde{J}$ is an element coming from an $\mathcal{A}$-space, we may use Proposition 3.20 and obtain "locality"

$$\begin{aligned}\psi(\Delta(D)\cdot E) &= (\Delta^{(4)}(D)\cdot \tilde{J}^{23}\cdot (i_+ \boxtimes i_-)(E))^{(23)}\\ &= \Delta^{(4)}(D)\cdot (\tilde{J}^{23}\cdot (i_+ \boxtimes i_-)(E))^{(23)}\\ &= \Delta^{(4)}(D)\cdot \psi(E)\,,\end{aligned}$$

as required.

We next compute $(\psi\boxtimes \mathrm{id})\psi(1_+\boxtimes 1_-)$ and $(\mathrm{id}\boxtimes\psi)\psi(1_+\boxtimes 1_-)$ in two ways. First, by the definition of ψ,

$$\begin{aligned}(\mathrm{id}\boxtimes\psi)\psi(1_+\boxtimes 1_-) &= ((1^{\boxtimes 2}\boxtimes \tilde{J}^{(23)})\cdot (\mathrm{id}^{\boxtimes 2}\boxtimes i_+\boxtimes i_-)(\tilde{J}\cdot(1_+\boxtimes 1_-)^{\boxtimes 2}))^{(45)}\\ &= (p_+\boxtimes p_-)^{\boxtimes 3}((1^{\boxtimes 2}\boxtimes\tilde{J})\cdot((\mathrm{id}^{\boxtimes 2}\boxtimes\Delta^{\boxtimes 2})(\tilde{J}))^{(45)})\,,\end{aligned}$$

and similarly

$$(\psi\boxtimes\mathrm{id})\psi(1_+\boxtimes 1_-) = (p_+\boxtimes p_-)^{\boxtimes 3}((\tilde{J}\boxtimes 1^{\boxtimes 2})\cdot((\Delta^{\boxtimes 2}\boxtimes\mathrm{id}^{\boxtimes 2})(\tilde{J}))^{(23)})\,.$$

Comparing with (5.10), we see that

$$\Delta^{\boxtimes 3}(\Phi)\cdot(\psi\boxtimes\mathrm{id})\psi(1_+\boxtimes 1_-) = (\mathrm{id}\boxtimes\psi)\psi(1_+\boxtimes 1_-)\,. \tag{5.12}$$

Second, by (5.11) we have

$$\begin{aligned}(\psi\boxtimes\mathrm{id})\psi(1_+\boxtimes 1_-) &= (\psi\boxtimes\mathrm{id})\phi^{\boxtimes 2}(J)\\ &= (\mathrm{id}^{\boxtimes 2}\boxtimes\phi)(\psi\phi\boxtimes\mathrm{id})(J)\,.\end{aligned}$$

Since both ψ and ϕ are $\vec{\mathcal{A}}$-maps,

$$\psi\phi(D) = \Delta^{(4)}(D)\cdot\psi\phi(1) = \Delta^{(4)}(D)\cdot\phi^{\boxtimes 2}(J) = \phi^{\boxtimes 2}(\Delta(D)\cdot J)\,,$$

for all $D\in\vec{\mathcal{A}}$. Therefore,

$$\begin{aligned}(\mathrm{id}^{\boxtimes 2}\boxtimes\phi)(\psi\phi\boxtimes\mathrm{id})(J) &= (\mathrm{id}^{\boxtimes 2}\boxtimes\phi)(\phi^{\boxtimes 2}\boxtimes\mathrm{id})((\Delta\boxtimes\mathrm{id})(J)\cdot J^{12})\\ &= \phi^{\boxtimes 3}((\Delta\boxtimes\mathrm{id})(J)\cdot J^{12})\,,\end{aligned}$$

so that

$$(\psi\boxtimes\mathrm{id})\psi(1_+\boxtimes 1_-) = \phi^{\boxtimes 3}((\Delta\boxtimes\mathrm{id})(J)\cdot J^{12})\,.$$

A similar computation shows that

$$(\mathrm{id}\boxtimes\psi)\psi(1_+\boxtimes 1_-) = \phi^{\boxtimes 3}((\mathrm{id}\boxtimes\Delta)(J)\cdot J^{23})\,.$$

Together with (5.12) we have

$$\Delta^{\boxtimes 3}(\Phi)\cdot\phi^{\boxtimes 3}((\Delta\boxtimes\mathrm{id})(J)\cdot J^{12}) = \phi^{\boxtimes 3}((\mathrm{id}\boxtimes\Delta)(J)\cdot J^{23})\,,$$

and using one more time that ϕ is an $\vec{\mathcal{A}}$-map, we finally get

$$\phi^{\boxtimes 3}(\Phi\cdot(\Delta\boxtimes\mathrm{id})(J)\cdot J^{12}) = \phi^{\boxtimes 3}((\mathrm{id}\boxtimes\Delta)(J)\cdot J^{23})\,.$$

The proof of (5.9) is now finished by applying the map $(\phi^{-1})^{\boxtimes 3}:(\mathcal{M}_+\boxtimes\mathcal{M}_-)^{\boxtimes 3}\to\vec{\mathcal{A}}^{\boxtimes 3}$.

The R-matrix of $\vec{\mathcal{A}}_{\mathrm{EK}}$ is given, according to (4.27), by the formula

$$R_{\mathrm{EK}} = (J^{-1})^{21}\cdot\exp(\tfrac{1}{2}$$  $$)\cdot J\,.$$

The expansion (5.8) of R_{EK} is then an immediate consequence of (5.7). □