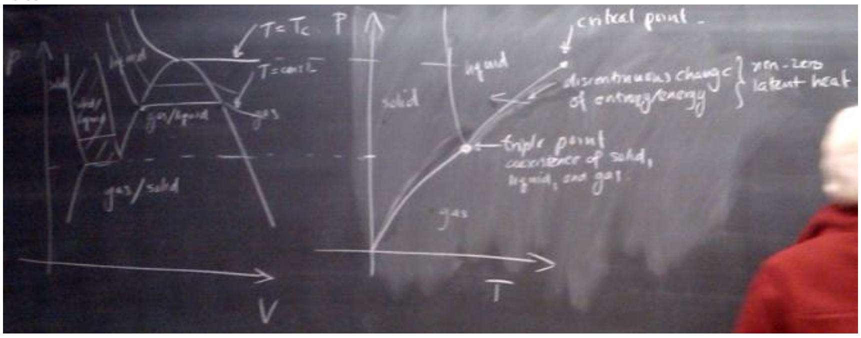
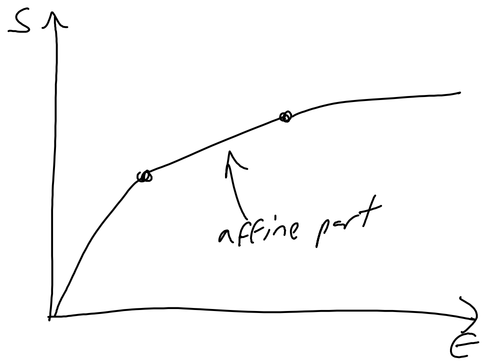
Lanford on StatMech, Jan 20 2010

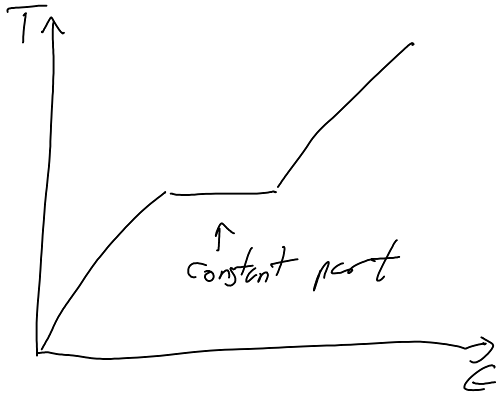
January-20-10
10:03 AM



<u>Question</u> How do such phase transitions manifest themselves in thermodynamic functions?

Assume v is held fixed





$$\frac{\partial S}{\partial E} = \frac{1}{T} \Rightarrow \begin{pmatrix}\text{affine pieces of} \\ \text{the } S/E \text{ graph}\end{pmatrix} \rightleftarrows \begin{pmatrix}\text{constant pieces} \\ \text{of the } T/E \text{ graph}\end{pmatrix}$$

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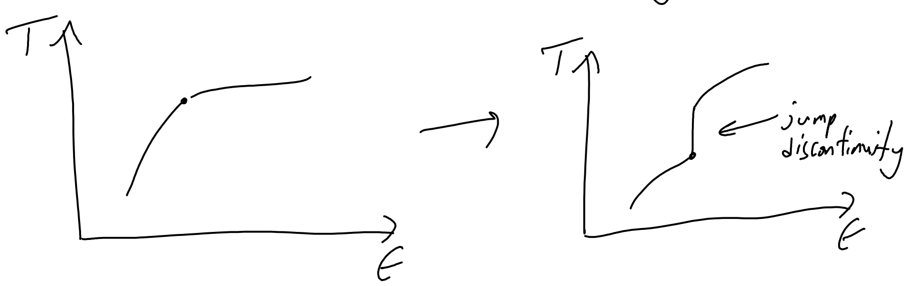
what are points of non-diffability of $E \to S(E)$



Does not seem to happen.

Hypothesis: $(E, v) \longmapsto S$ is everywhere diffable,

$\Rightarrow p, T$ are cont. functions of (E, v).

Helmholtz free energy: $F = E - TS$ or $f = \epsilon - Ts$

or, with $\beta = \frac{1}{T}$, $-\beta F = -\beta E + S$

or, with $\beta = \frac{1}{T}$, $-\beta F = -\beta E + S$

⋮

Proposition Let $u(t)$ be a convex function on $(a,b) \subset (0,\infty) \Longrightarrow s \mapsto s \cdot u(1/s)$ is also convex. Equivalently, $t \mapsto t u(t)$ convex $\Rightarrow \underset{\text{convex}}{s \mapsto u(\frac{1}{s})}$

More generally, if $u(t,\vec{x})$ is convex on a convex domain in $(0,\infty) \times \mathbb{R}^n$, then

$(s,\vec{y}) \longmapsto s\, u(\frac{1}{s}, \frac{\vec{y}}{s})$ is convex with a convex domain of definition.

Equivalently, $(t,\vec{x}) \mapsto t \cdot u(t,\vec{x})$ convex $\Rightarrow (s,\vec{y}) \mapsto u(s^{-1}, \vec{y})$ is convex.

pf

Follows from "$(t,\vec{z}) \longmapsto (\frac{1}{t}, \frac{\vec{z}}{t})$ carries lines into lines".

Let $u(\xi)$ be a convex function of one variable

set (Legendre transform)

$$\hat{u}(\eta) = \sup_{\xi} \{\xi\eta - u(\xi)\}$$ is again complex

then $u \mapsto \hat{u}$ is an involution.

(ξ^*, η^*) is a "conjugate pair" if

$\xi \longmapsto (\xi\eta^* - u(\xi))$ takes its supremum at ξ^*

(morally, if $u'(\xi^*) = \eta^*$)

$\Longleftrightarrow \hat{u}(\eta^*) = \xi^*\eta^* - u(\xi^*)$

$\Longleftrightarrow \xi^*\eta^* - u(\xi^*) - \hat{u}(\eta^*)$

<u>Fact</u>: u is diff'able at ξ^* iff $\hat{u}$ is strictly convex at η^*.

... Can be generalized to $\xi \in V$, $\eta \in V^*$ when V is a v.s. & V^* its dual.

$\beta F \Leftrightarrow$ the Legendre transform of S.

phase transitions $\Leftrightarrow$ non-strict concavity of S

$\Leftrightarrow$ non-diff'abilities of F.