**Claim 1.** No vertex is in the hull of any others.

2. The faces of $P_n$ correspond to ordered partitions of $[n] = \{1, \ldots, n\}$.

**Def.** If $(a_1, \ldots, a_n)$ is a permutation of $[n]$, define $(a_1, \ldots, a_{i-1}, a_i, \ldots, a_{i+1}, \ldots, a_n)$ to be the convex hull of the permutations in which

$$\forall l \in \{a_i, \ldots, a_{i+1}\} \notin \{a_i, \ldots, a_{i+1}\}$$

**E.g.:** $(3, 2, 4) = \text{Conv} \{(1,3,2,4), (1,1,3,4), (2,1,3,4)\}$

$$= (1,3,2,4) - (3,1,2,4)$$

Let $V = (a_1)(a_2) \ldots (a_n) = a_1 \ldots a_n$

**Def.** $(a_ia_{i+1})$ is the oriented segment from $V$ to $a_1 \ldots a_{i+1} a_i \ldots a_n$.

"The basic inward vectors at $V$" (**biv**)

as a vector $(a_i a_{i+1})$ is

$$[0, \ldots, 0, i+1, \ldots, -1, \ldots, 0]$$

**Example.**

- $2134 = (1234)$
- $1243 = (1243)$
- $2143 = (2143)$
Lemma The vectors \((\mathbf{a}_i, \mathbf{a}_j)\) for \(1 \leq i < j \leq n\), 
\[
(0, \ldots, 1, \ldots, -1, \ldots) = c_{a_i} - c_{a_j}
\]
are in the non-negative integer span of the basic inward vectors.

Lemma All vertices in \(P^n\) are in the non-negative integer span of the bivs at \(V\).

Proof on board.

Corollary No vertex of \(P^n\) is in the convex hull of others.

Note The biv make a basis for \(\mathbb{R}^{n-1} \cong P^n\) 
\[
= \{ (x) \in \mathbb{R}^n : \sum x_i = 1 \}
\]

Note Any \((n-2)\) biv at \(V\) defines a hyperplane \(H\) in \(\mathbb{R}^n\) so \(P^n\) lies entirely on one side of \(H\) (corresponding to the direction of the remaining biv).

Def Let \(H_i^{V+}\) be the hyperplane given by all bivs at \(V\) other than \((\mathbf{a}_i, \mathbf{a}_j)\), and \(H_i^{V+}\) the corresponding non-negative span.

Note \(H_i^{V+} \cap P^n\) is on the body of \(P^n\)

Claim \(H_i^{V+} \cap P^n = (a_1, \ldots, a_i) (a_{i+1}, \ldots, a_n)\)

PF on board.

Def Write \(H_{i\ldots i\ldots i}\) for the subspace generated by the biv at \(V\) other than \((\mathbf{a}_i, \mathbf{a}_j)\)
This image shows a weight diagram for the representation whose highest weight is "(1,1,1)," that is, the sum of the three fundamental weights. What we have here is just the orbit of the highest weight under the Weyl group (24 elements). These form the vertices of the "weight polyhedron" for this representation.

Pasted from [http://www.nd.edu/~bhall/book/a3wtdiag1.html](http://www.nd.edu/~bhall/book/a3wtdiag1.html)