General question: Given an autom. $\pi$, what can we say about $L(1, \pi)$ or more generally, about $L(k, \pi)$, $k \in \mathbb{Z}$.

Dirichlet: Given $\chi: (\mathbb{Z}/\mathbb{Z})^* \to \mathbb{C}^*$

set $L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$

Euler: $\zeta(2k) \notin \mathbb{T}^{2k} \mathbb{Q}^*$, but little is known about $\zeta(2k+1)$.

Similarly, $L(2k, \chi)$ when $\chi(-1) = 1$

and $L(2k+1, \chi)$ when $\chi(-1) = -1$ are controlled, but transcendentality of the rest is open.

Let $K$ be an imaginary quadratic field, let $\chi$ be a character of the $F$-ideal class group of $\mathcal{O}_K$. We wish to study $L(1, \chi)$.

Definitions: Given an algebraic field K,

$\mathcal{O}_K$: The ring of integers in K.

It is a Dedekind domain - integral domain

with a unique factorization property for ideals:

Given an ideal $F$ in $\mathcal{O}_K$, let $a \sim b$ for ideals $a$ & $b$, if $(\alpha) a = (\beta) b$ for some $\alpha, \beta$ with $\alpha, \beta \in F$.

Call the group of equiv. classes $\mathcal{O}_K(F)$.

For $\chi \in \mathcal{O}_K(F)$, what can we say about...
$L(1, \chi)^2$

Theorem 1. If $K$ is imaginary quadratic & $F = \mathbb{Q}$, let $\chi \neq 1$. Then $L(1, \chi)/\mathbb{Q}$ is a $\mathbb{Q}$-lin. comb. of alg. numbers. (hence it is transcendental.)

2. The $L(1, \chi)$'s are lin. indep. over $\mathbb{Q}$.

Theorem 2. With $K$ as before & $F \neq \mathbb{Q}$, then the $L(1, \chi)$'s are lin. indep. over $\mathbb{Q}$ and ... same...