1. Make sure I have many puzzles available.
2. Rewrite or remove the part about Gaussian elimination.
3. Fix left-right issues in handout and program.
4. Produce a program that will keep track of tricks and their lengths.
5. Find a very compact cube algorithm, put it on handout.
6. Bring the Seress book, extract a list of good problems from it.
7. Find a philosophical explanation for "the twist".

---

Dear Colleagues,

Non-Commutative Gaussian Elimination and Rubik's Cube

The Problem. Let \( G = \langle g_1, \ldots, g_n \rangle \) be a subgroup of \( S_n \), with \( n = O(100) \). Before you die, understand \( G \):
1. Compute \([G]\).
2. Given \( \sigma \in S_n \), decide if \( \sigma \in G \).
3. Write a \( \sigma \in G \) in terms of \( g_1, \ldots, g_n \).
4. Produce random elements of \( G \).

The Commutative Analog. Let \( V = \text{span}(v_1, \ldots, v_n) \) be a subspace of \( \mathbb{R}^n \). Before you die, understand \( V \).

Solution: Gaussian Elimination. Prepare an empty table.

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>\ldots</th>
<th>n-1</th>
<th>n</th>
</tr>
</thead>
</table>

Space for a vector \( u \in V \), of the form \( u = (0, 0, u_3, \ldots, u_n; 1) \): \( u_3 \) is the "pivot".

Feed \( v_1, \ldots, v_n \) in order. To feed a non-zero \( v \), find its pivotal position \( i \).
1. If box \( i \) is empty, put \( v \) there.
2. If box \( i \) is occupied, find a combination \( v' \) of \( v \) and \( u_i \) that eliminates the pivot, and feed \( v' \).

Non-Commutative Gaussian Elimination
Prepare a mostly-empty table.

<table>
<thead>
<tr>
<th>( 1 )</th>
<th>( 2 )</th>
<th>( 3 )</th>
<th>( \ldots )</th>
<th>( n )</th>
</tr>
</thead>
</table>

Space for a \( \sigma \in S_n \), of the form
\( (1, 2, \ldots, i-1, i-1, \ldots, 1, i, \ldots, n) \)

So \( \sigma \) fixes \( 1, \ldots, i-1 \), sends the "pivot" \( i \) to \( j \) and goes wild afterwards, and \( \sigma^{-1} \) does sticker \( j \).

Feed \( g_1, \ldots, g_n \) in order. To feed a non-identity \( \sigma \), find its pivotal position \( i \) and let \( j := \sigma(i) \).
1. If box \( (i, j) \) is empty, put \( \sigma \) there.
2. If box \( (i, j) \) contains \( \sigma \), feed \( \sigma' := \sigma^{i,j} \).

Theorem. \( G = M_1 \).
\( G = M_1 := \{ \sigma_{1,j} \sigma_{2,k} \cdots \sigma_{q,n} : \forall i, j \geq 1 \text{ and } \sigma_{i,k} \in T \} \).

Proof. The inclusions \( M_i \subseteq G \) and \( \{g_1, \ldots, g_n\} \subseteq M_i \) are obvious. The rest follows from the following
Lemma. \( M_i \) is closed under multiplication.

Proof. By backwards induction. Let
\( M_i := \{ \sigma_{1,j} \sigma_{2,k} \cdots \sigma_{q,n} : \forall i, j \geq 1 \text{ and } \sigma_{i,k} \in T \} \).

Clearly \( M_2 M_i \subseteq M_i \). Now assume that \( M_2 M_i \subseteq M_i \) and show that \( M_i M_i \subseteq M_i \).
Start with \( \sigma_{1,j} \sigma_{2,k} \epsilon M_i \):
\( \sigma_{1,j} \sigma_{2,k} \epsilon M_i \) \( \Rightarrow \sigma_{1,j} \sigma_{2,k} \epsilon M_i \) \( \Rightarrow \sigma_{1,j} \sigma_{2,k} \epsilon M_i \) \( \Rightarrow \sigma_{1,j} \sigma_{2,k} \epsilon M_i \).

Falls like a chain of dominos.

The Algorithm

A Demo Program

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
</table>

The Hardest Math I’ve Ever Really Used


Abstract. What’s the hardest math I’ve ever used in real life? Me, myself, directly — not by using a cellphone or a GPS device that somebody else designed. And in “real life” — not while studying or teaching mathematics?

I use addition and subtraction daily, adding up bills or calculating change. I use percentages often, though mostly it is just “add 15 percent.” I seldom use multiplication and division: when I buy in bulk, or when I need to know how many tiles I need to replace my kitchen floor. I’ve used powers twice in my life, doing calculations related to mortgages. I’ve used a tiny bit of 2 × 2 linear algebra for a tiny bit of non-math-related computer graphics I’ve played with.

And for a long time, that was it. In my talk I will tell you how recently a math topic discovered only in the 1800s made a brief and modest appearance in my non-mathematical life. There are many books devoted to that topic and a lot of active research. Yet for all I know, nobody ever needed the actual gory formulas for such a simple reason before.

...or an Art Historian... ...or an Environmentalist.

Goal. Find the least-blur path to go from Mona’s left eye to Mona’s right eye in fixed time. Alternatively, fix your blur-tolerance, and find the fastest path to do the same. For fixed blur, our camera moves at a speed proportional to its distance from the image plane:

The brachistochrome

\[ \frac{dx}{dt} = \frac{v}{\sqrt{2gy - x^2}} \quad \text{where} \quad v = \text{constant} \]

Bernoulli on Newton. “I recognize the line by his pace.”

The Happy Segway Principle

A Segway is happy iff both its wheels are happy unhappy

The Lobachevsky Space

Unhappy Segways Happy Segways

Two parallels through one point

The Actual Code

- \[ \theta^2(t) = \sin(\theta(t)) \]
- \[ \theta = 2 \text{arc} e^t \]

O-type word. - A. A. A. B. B.

Warning: Real numbers are imaginary. \[ \text{along with a code sample.} \]

\[ p_p ** p(a...) := p([a]) \]

means that my multiplication rule is like composition of functions.

2009-07 Page 2