

Non Commutative Gaussian Elimination - Program 5

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Amended from a similar notebook by Dror Bar-Natan and Itai Bar-Natan. The original version is at <http://www.math.toronto.edu/~drorbn/Misc/SchreierSimsRubik/>.

Pensieve Header: NCGE Program 5 - "TeaseFeed" is implemented with lovely results.

The Cube

The Generating Permutations

```
n = 54; $RecursionLimit = 2^16;
Generators = {
  M[{18, 27, 36, 4, 5, 6, 7, 8, 9, 3, 11, 12, 13, 14, 15, 16, 17,
    45, 2, 20, 21, 22, 23, 24, 25, 26, 44, 1, 29, 30, 31, 32, 33, 34, 35, 43,
    37, 38, 39, 40, 41, 42, 10, 19, 28, 52, 49, 46, 53, 50, 47, 54, 51, 48},
    {BottomFace}, 1],
  M[{1, 2, 3, 4, 5, 6, 16, 25, 34, 10, 11, 9, 15, 24, 33, 39, 17,
    18, 19, 20, 8, 14, 23, 32, 38, 26, 27, 28, 29, 7, 13, 22, 31, 37, 35, 36,
    12, 21, 30, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54},
    {TopFace}, 1],
  M[{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17,
    18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 31, 32, 33, 34, 35, 36, 48, 47, 46,
    39, 42, 45, 38, 41, 44, 37, 40, 43, 30, 29, 28, 49, 50, 51, 52, 53, 54},
    {FrontFace}, 1],
  M[{3, 6, 9, 2, 5, 8, 1, 4, 7, 54, 53, 52, 10, 11, 12, 13, 14,
    15, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36,
    37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 18, 17, 16},
    {BackFace}, 1],
  M[{13, 2, 3, 22, 5, 6, 31, 8, 9, 12, 21, 30, 37, 14, 15, 16,
    17, 18, 11, 20, 29, 40, 23, 24, 25, 26, 27, 10, 19, 28, 43, 32, 33, 34, 35,
    36, 46, 38, 39, 49, 41, 42, 52, 44, 45, 1, 47, 48, 4, 50, 51, 7, 53, 54},
    {LeftFace}, 1],
  M[{1, 2, 48, 4, 5, 51, 7, 8, 54, 10, 11, 12, 13, 14, 3, 18, 27,
    36, 19, 20, 21, 22, 23, 6, 17, 26, 35, 28, 29, 30, 31, 32, 9, 16, 25, 34,
    37, 38, 15, 40, 41, 24, 43, 44, 33, 46, 47, 39, 49, 50, 42, 52, 53, 45},
    {RightFace}, 1]
};
```

Program 5

```

Clear[s, M, T]; TC = 0;
M /: M[a1_, {w1___}, m1_] ** M[a2_, {w2___}, m2_] := M[a1[[a2]], {w1, w2}, m1 + m2];
M /: Inverse[M[a_, w_, m_]] := M[Ordering[a], -Reverse[w], m];
Feed[M[Range[n], __]] := {};
Feed[M[a_, {w___}, m_] := Module[
  {i, j, sij, k, l, skl},
  For[i = 1, a[[i]] == i, ++i]; j = a[[i]];
  If[Head[sij = s[i, j]] === Integer,
    (* then *) If[m ≥ T[sij][[3]],
      Feed[Inverse[ReplacePart[T[sij], {sij}, 2]] ** M[a, {w}, m]],
      T[s[i, j] = ++TC] = M[a, {w}, m];
      Feed[Inverse[M[a, {w}, m]] ** ReplacePart[T[sij], {sij}, 2]]
    ],
    (* else *) T[s[i, j] = ++TC] = M[a, {w}, m];
  Do[
    If[Head[skl = s[k, l]] == Integer,
      Feed[ReplacePart[T[sij] ** T[skl], {sij, skl}, 2]];
      Feed[ReplacePart[T[skl] ** T[sij], {skl, sij}, 2]]
    ],
    {k, n}, {l, n}
  ]
];
Images[i_] := Prepend[Select[Range[n], Head[s[i, #]] === Integer &], i];
MoveCount[i_, i_] := 0;
MoveCount[i_, j_] := T[s[i, j]][[3]];
TMC[] := Sum[Total[MoveCount[i, #] & /@ Images[i]], {i, n}];
Optimize[] := Timing[
  Do[
    If[Head[sij = s[i, j]] == Integer, Do[
      If[Head[skl = s[k, l]] == Integer,
        Feed[ReplacePart[T[sij] ** T[skl], {sij, skl}, 2]]
      ], {k, n}, {l, n}]],
    {i, n}, {j, n}];
  TMC[]
];

```

```

gens = Rest[Generators];
g = 0;
Print[Timing[
  (++)g; Feed[#]; Product[Length[Images[i]], {i, n}]] & /@ Join[gens, Inverse /@ gens]
]];
Print[tmc = TMC[]];
While[
  Last[opt = Optimize[]] ≠ tmc,
  tmc = Last[opt];
  Print[opt]
]
{125.705, {4, 73 483 200, 159 993 501 696 000, 1 802 166 803 103 744 000,
  43 252 003 274 489 856 000, 43 252 003 274 489 856 000, 43 252 003 274 489 856 000,
  43 252 003 274 489 856 000, 43 252 003 274 489 856 000, 43 252 003 274 489 856 000}}
14078
{115.987, 1731}
{116.876, 1516}
{115.316, 1509}

```

The Worst Case Scenario

```

Sum[Max[MoveCount[i, #] & /@ Images[i]], {i, n}]
206

```

And finally - one cube really solved...

```

P /: P[a1___] ** P[a2___] := P[a1][[{a2}]];
P /: Inverse[P[a___]] := P@@Ordering[{a}];
TeaseFeed[P@@Range[n]] := {};
TeaseFeed[p_P] := Module[
  {i, j, sij},
  For[i = 1, p[[i]] == i, ++i; j = p[[i]];
  If[Head[sij = s[i, j]] === Integer,
    Prepend[TeaseFeed[Inverse[P@@T[sij][[1]]] ** p], sij],
    {p}
  ]
];
P@@Range[54]
P[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18,
  19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36,
  37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54]

```

```
TeaseFeed[P[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16,
  17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35,
  36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54]]
{}
```

```
TeaseFeed[P[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16,
  17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35,
  36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 54, 53]]
{P[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17,
  18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36,
  37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 54, 53]}
```

```
gen1 = P@@Generators[[1, 1]]
```

```
P[18, 27, 36, 4, 5, 6, 7, 8, 9, 3, 11, 12, 13, 14, 15, 16, 17,
  45, 2, 20, 21, 22, 23, 24, 25, 26, 44, 1, 29, 30, 31, 32, 33, 34, 35, 43,
  37, 38, 39, 40, 41, 42, 10, 19, 28, 52, 49, 46, 53, 50, 47, 54, 51, 48]
```

```
TeaseFeed[gen1]
```

```
{5562, 5681, 5820, 5498, 6042, 5504, 6093, 6195, 6177, 6150, 6209, 6224, 6169}
```

```
TeaseFeed[gen1] /. t_Integer => T[t][[2]]
```

```
{{4622, 4994}, {-2569, 1, 5066}, {-5066, 2, 5372}, {2538, 4985},
  {4, 5856}, {-5, -4622, 2538, 4622}, {-5709, 4981, 5698}, {4, 6082},
  {-6018, 5, 4622}, {-4, -4983, 6146, 6079}, {-4981, -6018, 6162, 4622},
  {-6168, -6213, -4622, 6213, 4622}, {-4981, -6053, -6123, -6131, 6091, 4622}}
```

```
TeaseFeed[gen1] /. t_Integer => T[t][[2]] /. {
  t_Integer /; t > 0 => T[t][[2]],
  t_Integer /; t < 0 => -Reverse[T[-t][[2]]]
}
```

```
{{{RightFace}, {4982, 2569}}, {{-LeftFace}, {TopFace}, {2569, 4982}},
  {{-4982, -2569}, {1, 1}, {4985, 966}}, {{-966, 1, 966}, {-RightFace}},
  {{FrontFace}, {-5684, 5221, 1}}, {{-1, -4, 1}, {-RightFace}, {-966, 1, 966}, {RightFace}},
  {{-5373, -1}, {-FrontFace}, {-4622, 1, 4622}},
  {{FrontFace}, {-4985, -4981, 4985, 5763}}, {{-4622, -4}, {-1, 4, 1}, {RightFace}},
  {{-FrontFace}, {LeftFace}, {1, 6026}, {4981, 4983}},
  {{FrontFace}, {-4622, -4}, {-6092, 6081, 4980}, {RightFace}},
  {{-6018, -6091, 4622, 4}, {-6146, -4981}, {-RightFace}, {4981, 6146}, {RightFace}},
  {{FrontFace}, {-6052, -4}, {-6085, -4985, 6014},
  {-5691, -4622}, {-1, 4981, 1}, {RightFace}}}
```

```

FixedPoint[
  Flatten[Replace[#, {
    t_Integer /; t > 0 => T[t][[2]],
    t_Integer /; t < 0 => -Reverse[T[-t][[2]]]
  }, {1}]] &,
  TeaseFeed[gen1]
]
{RightFace, -BackFace, LeftFace, -LeftFace, TopFace, LeftFace, -BackFace, BackFace,
-LeftFace, TopFace, TopFace, -RightFace, BackFace, -BackFace, TopFace, BackFace,
-RightFace, FrontFace, LeftFace, -TopFace, -LeftFace, -BackFace, -LeftFace, BackFace,
TopFace, -TopFace, -FrontFace, TopFace, -RightFace, -BackFace, TopFace, BackFace,
RightFace, BackFace, RightFace, -BackFace, -TopFace, -FrontFace, -RightFace,
TopFace, RightFace, FrontFace, RightFace, FrontFace, -RightFace, TopFace, BackFace,
-RightFace, -BackFace, -TopFace, -RightFace, -FrontFace, -TopFace, FrontFace,
TopFace, RightFace, -FrontFace, LeftFace, TopFace, FrontFace, -TopFace, -FrontFace,
-LeftFace, FrontFace, -RightFace, -FrontFace, TopFace, FrontFace, RightFace,
-FrontFace, -RightFace, -TopFace, RightFace, -RightFace, -FrontFace, -TopFace,
FrontFace, TopFace, RightFace, FrontFace, TopFace, -FrontFace, -TopFace, FrontFace,
-RightFace, -FrontFace, TopFace, FrontFace, -TopFace, RightFace, FrontFace, -TopFace,
-TopFace, BackFace, RightFace, -BackFace, -RightFace, -TopFace, RightFace, -TopFace,
-FrontFace, -FrontFace, -RightFace, FrontFace, RightFace, FrontFace, TopFace,
-RightFace, RightFace, -TopFace, -RightFace, -TopFace, -FrontFace, TopFace, RightFace}

```

```

TeaseFeed[gen1] //. {
  -x_ => -x,
  t_Integer /; t > 0 => T[t][[2]],
  t_Integer /; t < 0 => -Reverse[T[-t][[2]]]
}
{{{RightFace}, {{-BackFace}, {LeftFace}}},
{{-LeftFace}, {TopFace}, {{LeftFace}, {-BackFace}}},
{{{BackFace}, {-LeftFace}}, {{TopFace}, {TopFace}}, {{-RightFace}, {BackFace}}},
{{{BackFace}, {TopFace}, {BackFace}}, {-RightFace}},
{{FrontFace}, {{{LeftFace}, {-TopFace}, {-LeftFace}},
  {{-BackFace}, {-LeftFace}, {BackFace}}, {TopFace}}},
{{{TopFace}, {-FrontFace}, {TopFace}}, {-RightFace},
  {{-BackFace}, {TopFace}, {BackFace}}, {RightFace}},
{{{BackFace}, {RightFace}, {-BackFace}}, {-TopFace}}, {-FrontFace},
  {{-RightFace}, {TopFace}, {RightFace}}}, {{FrontFace}, {{RightFace}, {FrontFace},
  {-RightFace}}, {{{TopFace}, {{BackFace}, {-RightFace}, {-BackFace}}}, {-TopFace}}},
{{{RightFace}, {-FrontFace}}, {{-TopFace}, {FrontFace}, {TopFace}}, {RightFace}},
{{-FrontFace}, {LeftFace}, {{TopFace}, {{FrontFace}, {-TopFace}}},
  {{-FrontFace}, {-LeftFace}}}, {{FrontFace}, {{-RightFace}, {-FrontFace}},
  {{{TopFace}, {FrontFace}}, {{RightFace}, {-FrontFace}, {-RightFace}}, {-TopFace}},
  {RightFace}}, {{{-RightFace}, {-FrontFace}}, {{-TopFace}, {FrontFace}, {TopFace}},
  {RightFace}, {FrontFace}}, {{{TopFace}, {-FrontFace}}, {-TopFace}}, {FrontFace}},
  {-RightFace}, {{-FrontFace}, {{TopFace}, {{FrontFace}, {-TopFace}}}, {RightFace}},
  {{FrontFace}, {{{-TopFace}, {-TopFace}}, {{BackFace}, {RightFace}, {-BackFace}}},
  {{-RightFace}, {-TopFace}, {RightFace}}, {-TopFace}}, {-FrontFace}}, {-FrontFace}},
  {{{-RightFace}, {FrontFace}}, {RightFace}, {{FrontFace}, {{TopFace}, {-RightFace}}},
  {{{RightFace}, {-TopFace}}, {-RightFace}},
  {{-TopFace}, {-FrontFace}, {TopFace}}, {RightFace}}}

```

```

Flatten[
  TeaseFeed[gen1] //. {
    -x_ => -x,
    t_Integer /; t > 0 => T[t][[2]],
    t_Integer /; t < 0 => -Reverse[T[-t][[2]]]
  }
]
{RightFace, -BackFace, LeftFace, -LeftFace, TopFace, LeftFace, -BackFace, BackFace,
 -LeftFace, TopFace, TopFace, -RightFace, BackFace, -BackFace, TopFace, BackFace,
 -RightFace, FrontFace, LeftFace, -TopFace, -LeftFace, -BackFace, -LeftFace, BackFace,
 TopFace, -TopFace, -FrontFace, TopFace, -RightFace, -BackFace, TopFace, BackFace,
 RightFace, BackFace, RightFace, -BackFace, -TopFace, -FrontFace, -RightFace,
 TopFace, RightFace, FrontFace, RightFace, FrontFace, -RightFace, TopFace, BackFace,
 -RightFace, -BackFace, -TopFace, -RightFace, -FrontFace, -TopFace, FrontFace,
 TopFace, RightFace, -FrontFace, LeftFace, TopFace, FrontFace, -TopFace, -FrontFace,
 -LeftFace, FrontFace, -RightFace, -FrontFace, TopFace, FrontFace, RightFace,
 -FrontFace, -RightFace, -TopFace, RightFace, -RightFace, -FrontFace, -TopFace,
 FrontFace, TopFace, RightFace, FrontFace, TopFace, -FrontFace, -TopFace, FrontFace,
 -RightFace, -FrontFace, TopFace, FrontFace, -TopFace, RightFace, FrontFace, -TopFace,
 -TopFace, BackFace, RightFace, -BackFace, -RightFace, -TopFace, RightFace, -TopFace,
 -FrontFace, -FrontFace, -RightFace, FrontFace, RightFace, FrontFace, TopFace,
 -RightFace, RightFace, -TopFace, -RightFace, -TopFace, -FrontFace, TopFace, RightFace}

cool = Flatten[
  TeaseFeed[gen1] //. {
    -x_ => -x,
    t_Integer /; t > 0 => T[t][[2]],
    t_Integer /; t < 0 => -Reverse[T[-t][[2]]]
  }
] //. {
  {Lft___, x_, -x_, rgt___} => {Lft, rgt},
  {Lft___, -x_, x_, rgt___} => {Lft, rgt},
  {Lft___, x_, x_, x_, rgt___} => {Lft, -x, rgt},
  {Lft___, -x_, -x_, -x_, rgt___} => {Lft, x, rgt}
}
{RightFace, -BackFace, -TopFace, -RightFace, TopFace, BackFace, -RightFace, FrontFace,
 LeftFace, -TopFace, -LeftFace, -BackFace, -LeftFace, BackFace, -FrontFace, TopFace,
 -RightFace, -BackFace, TopFace, BackFace, RightFace, BackFace, RightFace, -BackFace,
 -TopFace, -FrontFace, -RightFace, TopFace, RightFace, FrontFace, RightFace, FrontFace,
 -RightFace, TopFace, BackFace, -RightFace, -BackFace, -TopFace, -RightFace,
 -FrontFace, -TopFace, FrontFace, TopFace, RightFace, -FrontFace, LeftFace, TopFace,
 FrontFace, -TopFace, -FrontFace, -LeftFace, FrontFace, -RightFace, -FrontFace,
 TopFace, FrontFace, RightFace, -FrontFace, -RightFace, -TopFace, -FrontFace,
 -TopFace, FrontFace, TopFace, RightFace, FrontFace, TopFace, -FrontFace, -TopFace,
 FrontFace, -RightFace, -FrontFace, TopFace, FrontFace, -TopFace, RightFace,
 FrontFace, -TopFace, -TopFace, BackFace, RightFace, -BackFace, -RightFace,
 -TopFace, RightFace, -TopFace, -FrontFace, -FrontFace, -RightFace, FrontFace,
 RightFace, FrontFace, -RightFace, -TopFace, -FrontFace, TopFace, RightFace}

```

NonCommutativeMultiply @@

(cool /. Thread[{BottomFace, TopFace, FrontFace, BackFace, LeftFace, RightFace} →
Generators] /. -x_ := Inverse[x])

```
M[{18, 27, 36, 4, 5, 6, 7, 8, 9, 3, 11, 12, 13, 14, 15, 16, 17,
  45, 2, 20, 21, 22, 23, 24, 25, 26, 44, 1, 29, 30, 31, 32, 33, 34, 35, 43,
  37, 38, 39, 40, 41, 42, 10, 19, 28, 52, 49, 46, 53, 50, 47, 54, 51, 48},
{RightFace, -BackFace, -TopFace, -RightFace, TopFace, BackFace, -RightFace, FrontFace,
LeftFace, -TopFace, -LeftFace, -BackFace, -LeftFace, BackFace, -FrontFace, TopFace,
-RightFace, -BackFace, TopFace, BackFace, RightFace, BackFace, RightFace, -BackFace,
-TopFace, -FrontFace, -RightFace, TopFace, RightFace, FrontFace, RightFace, FrontFace,
-RightFace, TopFace, BackFace, -RightFace, -BackFace, -TopFace, -RightFace,
-FrontFace, -TopFace, FrontFace, TopFace, RightFace, -FrontFace, LeftFace, TopFace,
FrontFace, -TopFace, -FrontFace, -LeftFace, FrontFace, -RightFace, -FrontFace,
TopFace, FrontFace, RightFace, -FrontFace, -RightFace, -TopFace, -FrontFace,
-TopFace, FrontFace, TopFace, RightFace, FrontFace, TopFace, -FrontFace, -TopFace,
FrontFace, -RightFace, -FrontFace, TopFace, FrontFace, -TopFace, RightFace,
FrontFace, -TopFace, -TopFace, BackFace, RightFace, -BackFace, -RightFace,
-TopFace, RightFace, -TopFace, -FrontFace, -FrontFace, -RightFace, FrontFace,
RightFace, FrontFace, -RightFace, -TopFace, -FrontFace, TopFace, RightFace}, 97]
```

Generators[[1]]

```
M[{18, 27, 36, 4, 5, 6, 7, 8, 9, 3, 11, 12, 13, 14, 15, 16, 17, 45, 2,
  20, 21, 22, 23, 24, 25, 26, 44, 1, 29, 30, 31, 32, 33, 34, 35, 43, 37, 38, 39,
  40, 41, 42, 10, 19, 28, 52, 49, 46, 53, 50, 47, 54, 51, 48}, {BottomFace}, 1]
```

%%[[1]] == %%[[1]]

True