Non Commutative Gaussian
Elimination - Program 4

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Amended from a similar notebook by Dror Bar-Natan and Itai Bar-Natan. The original version is at http://www.math.toronto.edu/~drorbn/Misc/SchreierSimsRubik/.

Pensieve Header: NCGE Program 4 - replacing tricks with better ones when possible; at the end running "improvement sessions". The results are good.

The Cube
http://drorbn.net/AcademicPensieve/2009-07/#MathematicaNotebooks
The Generating Permutations

\[ n = 54; \ \$\text{RecursionLimit} = 2^{16}; \]
\[ \text{Generators} = \{ \]
\[ M[\{18, 27, 36, 4, 5, 6, 7, 8, 9, 3, 11, 12, 13, 14, 15, 16, 17, 45, 2, 20, 21, 22, 23, 24, 25, 26, 44, 1, 29, 30, 31, 32, 33, 34, 35, 43, 37, 38, 39, 40, 41, 42, 10, 19, 28, 52, 49, 46, 53, 50, 47, 54, 51, 48\}, \{\text{BottomFace}\}, 1], \]
\[ M[\{1, 2, 3, 4, 5, 6, 16, 25, 34, 10, 11, 9, 15, 24, 33, 39, 17, 18, 19, 20, 8, 14, 23, 32, 38, 26, 27, 28, 29, 7, 13, 22, 31, 37, 35, 36, 12, 21, 30, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54\}, \{\text{TopFace}\}, 1], \]
\[ M[\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 31, 32, 33, 34, 35, 36, 48, 47, 46, 39, 42, 45, 38, 41, 44, 37, 40, 43, 46, 47, 48, 49, 50, 51, 52, 53, 54\}, \{\text{FrontFace}\}, 1], \]
\[ M[\{3, 6, 9, 2, 5, 8, 1, 4, 7, 54, 53, 52, 10, 11, 12, 13, 14, 15, 16, 7, 31, 32, 33, 34, 35, 36, 46, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 54, 52, 53, 18, 17, 16\}, \{\text{BackFace}\}, 1], \]
\[ M[\{13, 2, 3, 22, 5, 6, 31, 8, 9, 12, 21, 30, 37, 14, 15, 16, 17, 18, 11, 20, 29, 40, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 46, 38, 39, 40, 41, 42, 43, 44, 45, 46, 1, 47, 48, 49, 50, 51, 7, 53, 54\}, \{\text{LeftFace}\}, 1], \]
\[ M[\{1, 2, 48, 4, 5, 51, 7, 8, 54, 10, 11, 12, 13, 14, 3, 18, 27, 36, 19, 20, 21, 22, 23, 6, 17, 26, 35, 28, 29, 30, 31, 32, 9, 16, 25, 34, 37, 38, 15, 40, 41, 24, 43, 44, 33, 46, 47, 39, 49, 50, 42, 52, 53, 45\}, \{\text{RightFace}\}, 1] \}
\]
Program 4

Clear[{s_M_T}; TC = 0;
M/: M[a1_, {w1___}, m1_] ** M[a2_, {w2___}, m2_] := M[a1[[a2]], {w1, w2}, m1 + m2];
M/: Inverse[M[a_, w_, m_]] := M[Ordering[a], -Reverse[w], m];
Feed[M[Range[n], __]] := Null;
Feed[M[a_, {w___}, m_]] := Module[
{i, j, sij, k, l, skl},
For[i = 1, a[[i]] = i, ++i; j = a[[i]];
If[Head[sij = s[i, j]] === Integer,
(* then *) If[m = T[sij][[3]],
Feed[ReplacePart[Inverse[T[sij]] ** M[a, {w}, m], {-sij, w}, 2]],
T[s[i, j] = ++TC] = M[a, {w}, m];
Feed[ReplacePart[Inverse[M[a, {w}, m]] ** T[sij], {-w, -sij}, 2]]
],
(* else *) T[s[i, j] = ++TC] = M[a, {w}, m];
Do[
If[Head[skl = s[k, l]] === Integer,
Feed[ReplacePart[T[sij] ** T[skl], {sij, skl}, 2]]; Feed[ReplacePart[T[skl] ** T[sij], {skl, sij}, 2]]
],
{k, n}, {l, n}
]
];
Images[i_] := Prepend[Select[Range[n], Head[s[i, #]] === Integer &], i];
MoveCount[i_, i_] := 0;
MoveCount[i_, j_] := T[s[i, j]][[3]]; 
TMC[] := Sum[Total[MoveCount[i, #] & /@ Images[i]], {i, n}];
Optimize[] := Timing[
Do[
If[Head[sij = s[i, j]] === Integer, Do[
If[Head[skl = s[k, l]] === Integer,
Feed[ReplacePart[T[sij] ** T[skl], {sij, skl}, 2]]
],
{k, n}, {l, n}],
{i, n}, {j, n}]; TMC[]
];
The Order of the Group

\[ g = 0; \]
\[
\text{Timing[}
\{++g; \text{Feed}[#]; \text{Product[Length[Images[i]}, \{i, n\}] & @ \}
\text{Join[Generators, Inverse /@ Generators]}
\}
\]
\{112.258, \{4, 16, 159993501696000, 21119142223872000, 43252003274489856000, 43252003274489856000, 43252003274489856000, 43252003274489856000, 43252003274489856000, 43252003274489856000, 43252003274489856000, 43252003274489856000\}\}

The Worst Case Scenario

\[
\text{Sum[Max[MoveCount[i, #] & @ Images[i]}, \{i, n\}]
\]
3089

\[
\text{Print[tmc = TMC[]];}
\]
\[
\text{While[}
\text{Last[opt = Optimize[]] \neq tmc,}
\text{tmc = Last[opt];}
\text{Print[opt]}
\]
14548
\{88.406, 1563\}
\{89.014, 1396\}
\{86.862, 1392\}

\[
\text{Sum[Max[MoveCount[i, #] & @ Images[i]}, \{i, n\}]
\]
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