Non Commutative Gaussian

Elimination - Program 4

By Dror Bar-Natan

Amended from a similar notebook by Dror Bar-Natan and Itai Bar-Natan. The original version is at http://www.math.toronto.edu/~drorbn/Misc/SchreierSimsRubik/.

Pensieve Header: NCGE Program 4 - replacing tricks with better ones when possible; at the end running "improvement sessions". The results are good.
The Cube

The Generating Permutations

\[ n = 54; \quad \text{\$RecursionLimit} = 2^{16}; \]
\[ \text{Generators} = \{ \]
\[ M[\{18, 27, 36, 4, 5, 6, 7, 8, 9, 3, 11, 12, 13, 14, 15, 16, 17, \]
\[ 45, 2, 20, 21, 22, 23, 24, 25, 26, 44, 1, 29, 30, 31, 32, 33, 34, 35, 43, \]
\[ 37, 38, 39, 40, 41, 42, 10, 19, 28, 52, 49, 46, 53, 50, 47, 54, 51, 48}, \]
\[ \{\text{BottomFace}\}, 1], \]
\[ M[\{1, 2, 3, 4, 5, 6, 16, 25, 34, 10, 11, 9, 15, 24, 33, 39, 17, \]
\[ 18, 19, 20, 8, 14, 23, 32, 38, 26, 27, 28, 29, 7, 13, 22, 31, 37, 35, 36, \]
\[ 12, 21, 30, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54\}, \]
\[ \{\text{TopFace}\}, 1], \]
\[ M[\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, \]
\[ 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 31, 32, 33, 34, 35, 36, 48, 47, 46, \]
\[ 39, 42, 45, 38, 41, 44, 37, 40, 43, 30, 29, 28, 49, 50, 51, 52, 53, 54\}, \]
\[ \{\text{FrontFace}\}, 1], \]
\[ M[\{3, 6, 9, 2, 5, 8, 1, 4, 7, 54, 53, 52, 10, 11, 12, 13, 14, \]
\[ 15, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, \]
\[ 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 17, 16\}, \]
\[ \{\text{BackFace}\}, 1], \]
\[ M[\{13, 2, 3, 22, 5, 6, 31, 8, 9, 12, 21, 30, 37, 14, 15, 16, \]
\[ 17, 18, 11, 20, 29, 40, 23, 24, 25, 26, 27, 10, 19, 28, 43, 32, 33, 34, 35, \]
\[ 36, 46, 38, 39, 49, 41, 42, 52, 44, 45, 1, 47, 48, 4, 50, 51, 7, 53, 54\}, \]
\[ \{\text{LeftFace}\}, 1], \]
\[ M[\{1, 2, 48, 4, 5, 51, 7, 8, 54, 10, 11, 12, 13, 14, 3, 18, 27, \]
\[ 36, 19, 20, 21, 22, 23, 6, 17, 26, 35, 28, 29, 30, 31, 32, 9, 16, 25, 34, \]
\[ 37, 38, 15, 40, 41, 24, 43, 44, 33, 46, 47, 39, 49, 50, 42, 52, 53, 45\}, \]
\[ \{\text{RightFace}\}, 1] \};
Program 4

```
Clear[{s, M, T}]; TC = 0;
M /: M[a1_, {w1___}, m1_] ** M[a2_, {w2___}, m2_] := M[a1[[a2]], {w1, w2}, m1 + m2];
M /: Inverse[M[a_, w_, m_]] := M[Ordering[a], -Reverse[w], m];
Feed[M[Range[n], __]] := Null;
Feed[M[a_, {w___}, m_]] := Module[
  {i, j, si, k, l, skl},
  If[Head[si = s[i, j]] === Integer,
    (* then *) If[m >= T[si][[3]],
      Feed[ReplacePart[Inverse[T[si]] ** M[a, {w}, m], {si, w}, 2]],
      T = T + TC = M[a, {w}, m];
      Feed[ReplacePart[Inverse[M[a, {w}, m]] ** T[si], -{w, -si}, 2] ]],
    (* else *) T[si, j] = TC = M[a, {w}, m];
  Do[
    If[Head[skl = s[k, l]] === Integer,
      Feed[ReplacePart[T[si] ** T[skl], {si, skl}, 2]]; 
      Feed[ReplacePart[T[skl] ** T[si], {skl, si}, 2]]
    , 
    {k, n}, {l, n}
  ]
  ];
Images[i] := Prepend[Select[Range[n], Head[s[i, #]] === Integer &], i];
MoveCount[i, _] := 0;
MoveCount[i, j] := T[si, j][[3]];
TMC[] := Sum[Total[MoveCount[i, #] & /@ Images[i]], {i, n}];
Optimize[] := Timing[
  Do[
    If[Head[si = s[i, j]] === Integer, Do[
      If[Head[skl = s[k, l]] === Integer,
        Feed[ReplacePart[T[si] ** T[skl], {si, skl}, 2]]
      , 
      {k, n}, {l, n}]
    , 
    {i, n}, {j, n}];
    TMC[]
  ];
];
```

The Order of the Group

\[
g = 0;
\text{Timing[}
\{++g; \text{Feed[#]; \text{Product}[\text{Length}[\text{Images}[[i]], \{i, n\}]] & @
\text{Join}[\text{Generators}, \text{Inverse} /@ \text{Generators}]
\}
\{112.258, \{4, 16, 159\ 993\ 501\ 696\ 000,
21\ 119\ 142\ 223\ 872\ 000, 43\ 252\ 003\ 274\ 489\ 856\ 000, 43\ 252\ 003\ 274\ 489\ 856\ 000,
43\ 252\ 003\ 274\ 489\ 856\ 000, 43\ 252\ 003\ 274\ 489\ 856\ 000, 43\ 252\ 003\ 274\ 489\ 856\ 000,
43\ 252\ 003\ 274\ 489\ 856\ 000, 43\ 252\ 003\ 274\ 489\ 856\ 000, 43\ 252\ 003\ 274\ 489\ 856\ 000\}\}
\]
\]

The Worst Case Scenario

\[
\text{Sum[Max[\text{MoveCount}[i, #] & @ \text{Images}[i]], \{i, n\}]}
3\ 089
\text{Print[tmc = \text{TMC}[];}
\text{While[}
\text{\text{Last[\text{opt} = \text{Optimize[]}]} \neq \text{tmc},}
\text{tmc = \text{Last[\text{opt}];}
\text{Print[\text{opt}]}
\text{]}
14\ 548
\{88.406, 1563\}
\{89.014, 1396\}
\{86.862, 1392\}
\text{Sum[Max[\text{MoveCount}[i, #] & @ \text{Images}[i]], \{i, n\}]}
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