Non Commutative Gaussian Elimination - Program 4

By Dror Bar-Natan

Amended from a similar notebook by Dror Bar-Natan and Itai Bar-Natan. The original version is at http://www.math.toronto.edu/~dorbn/Misc/SchreierSimsRubik/

Pensieve Header: NCGE Program 4 - replacing tricks with better ones when possible; at the end running "improvement sessions". The results are good.

The Cube
The Generating Permutations

\[ n = 54; \text{\$RecursionLimit = 2^16; } \]
\[
\text{Generators} = \{ \\
M[\{18, 27, 36, 4, 5, 6, 7, 8, 9, 3, 11, 12, 13, 14, 15, 16, 17, \\
45, 2, 20, 21, 22, 23, 24, 25, 26, 44, 1, 29, 30, 31, 32, 33, 34, 35, 43, \\
37, 38, 39, 40, 41, 42, 10, 19, 28, 52, 49, 46, 53, 50, 47, 54, 51, 48\}, \\
\{\text{BottomFace}, 1\}], \\
M[\{1, 2, 3, 4, 5, 6, 16, 25, 34, 10, 11, 9, 15, 24, 33, 39, \\
17, 18, 19, 20, 8, 14, 23, 32, 38, 26, 27, 28, 29, 7, 13, 22, 31, 37, 35, 36, \\
12, 21, 30, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54\}, \\
\{\text{TopFace}, 1\}], \\
M[\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, \\
18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 31, 32, 33, 34, 35, 36, 48, 47, 46, \\
39, 42, 45, 38, 41, 44, 37, 40, 43, 30, 29, 28, 49, 50, 51, 52, 53, 54\}, \\
\{\text{FrontFace}, 1\}], \\
M[\{3, 6, 9, 2, 5, 8, 1, 4, 7, 54, 53, 52, 10, 11, 12, 13, 14, \\
15, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, \\
37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 18, 17, 16\}, \\
\{\text{BackFace}, 1\}], \\
M[\{13, 2, 3, 22, 5, 6, 31, 8, 9, 12, 21, 30, 37, 14, 15, 16, \\
17, 18, 11, 20, 29, 40, 23, 24, 25, 26, 27, 10, 19, 28, 43, 32, 33, 34, 35, \\
36, 46, 38, 39, 49, 41, 42, 52, 44, 45, 41, 47, 48, 4, 50, 51, 7, 53, 54\}, \\
\{\text{LeftFace}, 1\}], \\
M[\{1, 2, 48, 4, 5, 51, 7, 8, 54, 10, 11, 12, 13, 14, 3, 18, \\
27, 36, 19, 20, 21, 22, 23, 6, 17, 26, 35, 28, 29, 30, 31, 32, 9, 16, 25, 34, \\
37, 38, 15, 40, 41, 24, 43, 44, 33, 46, 47, 39, 49, 50, 42, 52, 53, 45\}, \\
\{\text{RightFace}, 1\} \\
\}; \]
Program 4

Clear[s, M, T]; TC = 0;
M /: M[a1_, (w1___), m1_] := M[a1[[a2]], (w1, w2), m1 + m2];
M /: Inverse[M[a_, w_, m_]] := M[Ordering[a], -Reverse[w], m];
Feed[M[Range[n], ___]] := Null;
Feed[M[a_, (w___), m_]] := Module[
    {i, j, sj, k, l, skl},
    For[i = 1, a[[i]] = i, ++i; j = a[[i]]];
    If[Head[s] = s[i, j]] == Integer,
    (* then *) If[m ≥ T[sij][3]],
    Feed[ReplacePart[Inverse[T[sij]] ** M[a, (w), m], {-sij, w}, 2]],
    T[s[i, j] = ++TC] = M[a, (w), m];
    Feed[ReplacePart[Inverse[M[a, (w), m]] ** T[sij], {-w, -sij}, 2]]
    ],
    (* else *) T[s[i, j] = ++TC] = M[a, (w), m];
Do[
    If[Head[skl = s[k, 1]] == Integer,
    Feed[ReplacePart[T[sij] ** T[skl], {sij, skl}, 2]];]
    Feed[ReplacePart[T[skl] ** T[sij], {skl, sij}, 2]]
    ]
    ];
Images[i_] := Prepend[Select[Range[n], Head[s[i, #]] == Integer &], i];
MoveCount[i_, i_] := 0;
MoveCount[i_, j_] := T[s[i, j]][[3]];]
TMC[] := Sum[Total[MoveCount[i, #] & /@ Images[i]], (i, n)];
Optimize[] := Timing[
    Do[
    If[Head[s] = s[i, j]] == Integer, Do[
    If[Head[skl = s[k, 1]] == Integer,
    Feed[ReplacePart[T[s[i, j] ** T[skl]], {s[i, j], skl}, 2]]
    ], (k, n), {l, n}]],
    (i, n), {j, n}];
    TMC[]
    ];

- The Order of the Group

g = 0;
Timing[
  (+g; Feed[#]; Product[Length[Images[i]], (i, n)]) & /@
  Join[Generators, Inverse /@ Generators]
]
{112.258, {4, 16, 159993501696000, 2111914223872000, 43252003274489856000, 43252003274489856000, 43252003274489856000, 43252003274489856000, 43252003274489856000, 43252003274489856000, 43252003274489856000, 43252003274489856000, 43252003274489856000, 43252003274489856000, 43252003274489856000, 43252003274489856000}}
The Worst Case Scenario

\[
\text{Sum}[\text{Max}[\text{MoveCount}[i, #] \& \@ \text{Images}[i]], \{i, n\}]
\]

3089

\[
\text{Print}[tmc = \text{TMC}[]];
\]

While[
    \[
    \text{Last}[\text{opt} = \text{Optimize[]}]] \neq tmc,
    \]
    \[
    tmc = \text{Last}[\text{opt}];
    \]
    \[
    \text{Print}[\text{opt}]
    \]

14548

\{88.406, 1563\}

\{89.014, 1396\}

\{86.862, 1392\}

\[
\text{Sum}[\text{Max}[\text{MoveCount}[i, #] \& \@ \text{Images}[i]], \{i, n\}]
\]

207