Non Commutative Gaussian Elimination - Program 1

By Dror Bar-Natan

Amended from a similar notebook by Dror Bar-Natan and Itai Bar-Natan. The original version is at http://www.math.toronto.edu/~drorbn/Misc/SchreierSimsRubik/.

Pensieve Header: NCGE Program 1 - keeping track of lengths of tricks; the results are sad.

The Cube

The Generating Permutations

```
n = 54; $RecursionLimit = 2^16;
Generators = {
   M[{18, 27, 36, 4, 5, 6, 7, 8, 9, 3, 11, 12, 13, 14, 15, 16, 17,
     45, 2, 20, 21, 22, 23, 24, 25, 26, 44, 1, 29, 30, 31, 32, 33, 34, 35, 43,
     37, 38, 39, 40, 41, 42, 10, 19, 28, 52, 49, 46, 53, 50, 47, 54, 51, 48},
    {BottomFace}, 1],
   M[{1, 2, 3, 4, 5, 6, 16, 25, 34, 10, 11, 9, 15, 24, 33, 39, 17,
     18, 19, 20, 8, 14, 23, 32, 38, 26, 27, 28, 29, 7, 13, 22, 31, 37, 35, 36,
     12, 21, 30, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54},
    {TopFace}, 1],
   M[{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17,
     18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 31, 32, 33, 34, 35, 36, 48, 47, 46,
     39, 42, 45, 38, 41, 44, 37, 40, 43, 30, 29, 28, 49, 50, 51, 52, 53, 54},
    {FrontFace}, 1],
   M[{3, 6, 9, 2, 5, 8, 1, 4, 7, 54, 53, 52, 10, 11, 12, 13, 14,
     15, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36,
     37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 18, 17, 16},
    {BackFace}, 1],
   M[{13, 2, 3, 22, 5, 6, 31, 8, 9, 12, 21, 30, 37, 14, 15, 16,
     17, 18, 11, 20, 29, 40, 23, 24, 25, 26, 27, 10, 19, 28, 43, 32, 33, 34, 35,
     36, 46, 38, 39, 49, 41, 42, 52, 44, 45, 1, 47, 48, 4, 50, 51, 7, 53, 54},
    {LeftFace}, 1],
   M[{1, 2, 48, 4, 5, 51, 7, 8, 54, 10, 11, 12, 13, 14, 3, 18, 27,
     36, 19, 20, 21, 22, 23, 6, 17, 26, 35, 28, 29, 30, 31, 32, 9, 16, 25, 34,
     37, 38, 15, 40, 41, 24, 43, 44, 33, 46, 47, 39, 49, 50, 42, 52, 53, 45},
    {RightFace}, 1]
  };
```

Program 1

```
Clear[s, M];
M /: M[a1_, {w1___}, m1_] ** M[a2_, {w2___}, m2_] := M[a1[[a2]], {w1, w2}, m1 + m2];
M /: Inverse[M[a_, w_, m_]] := M[Ordering[a], -Reverse[w], m];
Feed[M[Range[n], __]] := Null;
Feed[M[a_, {w___}, m_]] := Module[{i, j, k, l},
   For[i = 1, a[[i]] == i, ++i]; j = a[[i]];
   If[Head[s[i, j]] === M,
    Feed[ReplacePart[Inverse[s[i, j]] ** M[a, {w}, m], {-S[i, j], w}, 2]],
    s[i, j] = M[a, {w}, m];
    Do[
     If[Head[s[k, l]] == M,
      Feed[ReplacePart[s[i, j] ** s[k, l], {S[i, j], S[k, l]}, 2]];
      Feed[ReplacePart[s[k, l] ** s[i, j], {S[k, l], S[i, j]}, 2]]
     ],
     {k, n}, {l, n}
    ]
   ]
  ];
Images[i_] := Prepend[Select[Range[n], Head[s[i, #]] === M &], i]
```

The Order of the Group

```
Timing[
  (Feed[#]; Product[Length[Images[i]], {i, n}]) & /@ Generators
]
```

```
{103.865, {4, 16, 159 993 501 696 000,
  21 119 142 223 872 000, 43 252 003 274 489 856 000, 43 252 003 274 489 856 000}}
```

It is lovely to note that the number computed right above, the order of the Rubik's cube group according to our computer, agrees with the number that appears in the literature, for example, in Wikipedia. Note also that according to our computation the last generator of the group was actually not necessary.

The Worst Case Scenario

```
Sum[Max[Last[s[i, #]] & /@ Images[i]], {i, n}]
```

```
1 059 923 953 939 810
```

For the Patient : A Recipe for Solving the Cube

```
Reap[
   Do[If[Head[s[i, j]] === M, Sow[S[i, j] → Rest[s[i, j]]]], {i, n - 1}, {j, i + 1, n}]
  ][[2, 1]] // ColumnForm
```

```
S[1, 3] → M[{S[3, 37], S[1, 31]}, 2 977 037]
S[1, 7] → M[{S[3, 7], S[1, 3]}, 8 947 611]
S[1, 9] → M[{S[3, 9], S[1, 3]}, 20 864 120]
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S[1, 10] → M[{S[1, 18], S[1, 43]}, 17]
S[1, 12] → M[{S[7, 15], S[1, 15]}, 490 330 168 473]
S[1, 13] → M[{S[7, 13], S[1, 7]}, 199 899 209]
S[1, 15] → M[{S[3, 15], S[1, 3]}, 274 762 218]
S[1, 16] → M[{S[3, 16], S[1, 3]}, 1 550 614 226]
S[1, 18] → M[{BottomFace}, 1]
S[1, 28] → M[{S[1, 45], S[1, 18]}, 3]
S[1, 30] → M[{S[1, 37], S[1, 10]}, 7782]
S[1, 31] → M[{S[1, 36], S[1, 18]}, 5]
S[1, 33] → M[{S[1, 54], S[1, 3]}, 5 954 091]
S[1, 34] → M[{S[1, 10], S[1, 46]}, 16 471]
S[1, 36] → M[{S[3, 48], S[1, 18]}, 4]
S[1, 37] → M[{S[30, 31], S[1, 31]}, 7765]
S[1, 39] → M[{S[3, 33], S[1, 18]}, 15]
S[1, 43] → M[{S[1, 39], S[1, 18]}, 16]
S[1, 45] → M[{S[1, 18], S[1, 18]}, 2]
S[1, 46] → M[{S[3, 28], S[1, 18]}, 16 454]
S[1, 48] → M[{S[3, 45], S[1, 18]}, 52]
S[1, 52] → M[{S[1, 10], S[1, 10]}, 34]
S[1, 54] → M[{S[1, 3], S[1, 10]}, 2 977 054]
S[2, 4] → M[{-S[1, 3], S[4, 38], S[1, 3]}, 50 651 388]
S[2, 6] → M[{-S[1, 3], BackFace}, 2 977 038]
S[2, 8] → M[{-S[1, 7], S[24, 29], S[1, 7]}, 167 102 274]
S[2, 11] → M[{S[4, 6], S[2, 47]}, 192 121 876 680 667]
S[2, 14] → M[{S[6, 14], S[2, 6]}, 86 563 299 147 907]
S[2, 17] → M[{S[4, 17], S[2, 4]}, 192 424 693 921 820]
S[2, 19] → M[{-S[1, 10], S[19, 44], S[1, 10]}, 184]
S[2, 21] → M[{-S[1, 3], S[9, 36], S[1, 3]}, 51 440 227 323]
S[2, 22] → M[{S[21, 22], S[2, 21]}, 280 182 525 877]
S[2, 24] → M[{-S[1, 13], S[21, 51], S[1, 13]}, 191 814 538 594 760]
S[2, 25] → M[{-S[1, 18], S[21, 40], S[1, 18]}, 66 292 781 183]
S[2, 27] → M[{-S[1, 31], S[1, 18], S[1, 31]}, 11]
S[2, 29] → M[{S[2, 21], S[2, 24]}, 191 865 978 822 083]
S[2, 32] → M[{-S[1, 30], S[2, 22], S[1, 9]}, 280 203 397 779]
S[2, 35] → M[{-S[1, 31], S[2, 51], S[1, 54]}, 40 273 171 341 770]
S[2, 38] → M[{-S[1, 10], S[28, 33], S[1, 10]}, 50 960]
S[2, 40] → M[{-S[1, 10], S[28, 37], S[1, 10]}, 68 751]
S[2, 42] → M[{-S[1, 45], S[1, 48], S[1, 10]}, 71]
S[2, 44] → M[{-S[1, 10], S[27, 42], S[1, 10]}, 402]
S[2, 47] → M[{S[3, 9], S[2, 29]}, 191 865 996 709 166]
S[2, 49] → M[{-S[1, 7], S[27, 35], S[1, 7]}, 17 341 866 236 863]
S[2, 51] → M[{-S[1, 7], S[27, 47], S[1, 7]}, 40 273 168 364 711]
S[2, 53] → M[{-S[1, 3], S[35, 42], S[1, 3]}, 57 160 228 574]
S[3, 7] → M[{S[7, 16], S[3, 30]}, 5 970 574]
S[3, 9] → M[{-S[2, 42], -S[1, 3], S[9, 33], S[1, 3]}, 17 887 083]
S[3, 12] → M[{-S[2, 27], -S[1, 15], S[7, 9], S[1, 12]}, 1 642 087 560 492]
S[3, 13] → M[{-S[2, 42], -S[1, 3], S[7, 37], S[1, 3]}, 1 598 372 835]
S[3, 15] → M[{-S[1, 7], S[30, 33], S[1, 7]}, 271 785 181]
S[3, 16] → M[{-S[1, 3], S[9, 28], S[1, 3]}, 1 547 637 189]
S[3, 18] → M[{-S[2, 27], -S[1, 10], S[2, 42], S[1, 10]}, 116]
S[3, 28] → M[{-S[2, 27], -S[1, 18], S[27, 44], S[1, 18]}, 16 453]
S[3, 30] → M[{-S[2, 27], -S[1, 10], S[1, 33], S[1, 46]}, 5 970 573]
S[3, 31] → M[{-S[2, 19], -S[1, 36], S[1, 10], S[1, 34]}, 16 676]
S[3, 33] → M[{-S[1, 45], S[2, 27], S[1, 18]}, 14]
S[3, 34] → M[{-S[2, 19], -S[1, 45], S[3, 36], S[1, 18]}, 8002]
S[3, 36] → M[{-S[2, 27], -S[1, 31], S[1, 30], S[1, 10]}, 7815]
S[3, 37] → M[{-S[1, 18], S[28, 34], S[1, 18]}, 2 977 032]
S[3, 39] → M[{-S[1, 10], S[19, 40], S[1, 10]}, 17 217]
S[3, 43] → M[{-S[1, 10], S[19, 38], S[1, 10]}, 16 962]
S[3, 45] → M[{S[1, 52], S[1, 10]}, 51]
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S[3, 46] → M[{-S[2, 42], -S[1, 10], S[2, 44], S[1, 10]}, 507]
S[3, 48] → M[{-S[1, 18], S[28, 31], S[1, 18]}, 3]
S[3, 54] → M[{-S[1, 10], S[2, 19], S[1, 10]}, 218]
S[4, 6] → M[{-S[3, 28], -S[2, 6], -S[1, 7], S[4, 25], S[1, 7]}, 255 879 971 501]
S[4, 8] → M[{-S[3, 28], -S[2, 6], -S[1, 37], S[6, 42], S[1, 7]}, 68 096 886 284]
S[4, 11] → M[{-S[3, 36], -S[2, 6], S[2, 11], S[2, 4]}, 192 121 930 316 908]
S[4, 14] → M[{-S[3, 43], -S[2, 40], -S[1, 13], S[4, 24], S[1, 9]}, 41 585 830 285 675]
S[4, 17] → M[{S[4, 25], S[4, 51]}, 192 424 643 270 432]
S[4, 19] → M[{-S[3, 16], -S[2, 42], -S[1, 10], S[4, 42], S[1, 10]}, 257 660 264 146]
S[4, 21] → M[{-S[3, 48], -S[2, 6], S[4, 6], S[2, 4]}, 255 933 599 930]
S[4, 22] → M[{-S[2, 4], S[19, 22], S[2, 4]}, 41 305 367 271 378]
S[4, 24] → M[{-S[3, 37], -S[2, 32], -S[1, 7], S[4, 35], S[1, 7]}, 41 585 609 436 633]
S[4, 25] → M[{-S[3, 7], -S[2, 21], S[6, 21], S[2, 6]}, 255 859 082 788]
S[4, 27] → M[{-S[3, 13], -S[2, 38], -S[1, 18], S[2, 4], S[1, 7]}, 1 658 022 795]
S[4, 29] → M[{-S[3, 43], -S[1, 3], S[4, 32], S[1, 3]}, 1 765 499 851]
S[4, 32] → M[{-S[3, 16], -S[2, 38], -S[1, 3], S[4, 47], S[1, 3]}, 1 759 528 815]
S[4, 35] → M[{-S[1, 7], S[4, 22], S[1, 7]}, 41 305 385 166 600]
S[4, 38] → M[{-S[3, 37], -S[2, 6], -S[1, 3], S[4, 40], S[1, 3]}, 44 697 314]
S[4, 40] → M[{-S[3, 30], -S[2, 44], -S[1, 9], S[2, 6], S[1, 3]}, 32 789 170]
S[4, 42] → M[{-S[3, 7], -S[2, 8], -S[1, 3], S[4, 21], S[1, 3]}, 256 112 626 852]
S[4, 44] → M[{-S[3, 34], -S[2, 4], -S[1, 3], S[4, 27], S[1, 3]}, 1 714 636 259]
S[4, 47] → M[{-S[3, 28], -S[1, 3], S[4, 49], S[1, 3]}, 205 886 592]
S[4, 49] → M[{-S[3, 28], -S[2, 44], -S[1, 13], LeftFace}, 199 916 065]
S[4, 51] → M[{S[8, 22], S[4, 49]}, 192 168 784 187 644]
S[6, 8] → M[{-S[3, 33], -S[2, 19], -S[1, 7], S[6, 25], S[1, 7]}, 204 400 909 253]
S[6, 14] → M[{-S[3, 12], -S[2, 27], -S[1, 13], S[6, 22], S[1, 16]}, 86 563 296 170 869]
S[6, 17] → M[{-S[4, 27], -S[3, 7], -S[2, 42], -S[1, 13], S[2, 14], S[1, 9]}, 86 565 183 904 676]
S[6, 19] → M[{-S[4, 8], -S[3, 54], -S[2, 4], -S[1, 3], S[4, 8], S[1, 3]}, 136 250 378 248]
S[6, 21] → M[{-S[3, 33], -S[2, 38], -S[1, 7], S[6, 8], S[1, 39]}, 204 409 907 853]
S[6, 22] → M[{-S[3, 16], -S[2, 21], -S[1, 3], S[6, 24], S[1, 3]}, 84 919 458 096 931]
S[6, 24] → M[{-S[3, 36], -S[2, 21], -S[1, 37], S[6, 32], S[1, 28]}, 84 866 464 278 345]
S[6, 25] → M[{-S[3, 9], -S[1, 3], S[6, 40], S[1, 3]}, 204 383 013 833]
S[6, 27] → M[{-S[3, 45], -S[2, 25], -S[1, 39], S[6, 44], S[1, 7]}, 68 055 125 591]
S[6, 29] → M[{-S[3, 48], -S[2, 6], S[8, 49], S[2, 6]}, 84 221 726 945 830]
S[6, 32] → M[{-S[3, 30], -S[2, 21], -S[1, 9], S[6, 51], S[1, 9]}, 84 815 024 035 439]
S[6, 35] → M[{-S[3, 33], -S[2, 42], -S[1, 3], S[6, 47], S[1, 3]}, 84 221 744 824 707]
S[6, 38] → M[{-S[3, 28], -S[2, 42], -S[1, 3], S[6, 27], S[1, 3]}, 68 061 096 189]
S[6, 40] → M[{-S[4, 8], -S[3, 37], -S[2, 6], -S[1, 3], S[6, 19], S[1, 3]}, 204 359 172 676]
S[6, 42] → M[{-S[3, 9], -S[2, 42], -S[1, 3], S[6, 38], S[1, 3]}, 68 084 937 417]
S[6, 44] → M[{-S[4, 40], -S[3, 39], -S[2, 27], -S[1, 3], S[4, 44], S[1, 3]}, 1 753 396 731]
S[6, 47] → M[{-S[3, 37], -S[2, 42], -S[1, 36], S[6, 29], S[1, 7]}, 84 221 738 870 548]
S[6, 49] → M[{-S[3, 31], -S[2, 21], -S[1, 9], S[6, 35], S[1, 12]}, 84 763 536 101 299]
S[6, 51] → M[{-S[3, 36], -S[2, 19], -S[1, 45], S[6, 49], S[1, 28]}, 84 763 536 109 303]
S[7, 9] → M[{-S[3, 13], -S[2, 21], -S[1, 28], S[7, 31], S[1, 9]}, 1 151 482 629 790]
S[7, 12] → M[{-S[3, 9], -S[1, 7], S[8, 19], S[1, 7]}, 7 046 988 148]
S[7, 13] → M[{-S[1, 7], S[8, 40], S[1, 7]}, 190 951 598]
S[7, 15] → M[{-S[3, 13], S[9, 45], S[3, 13]}, 490 055 406 255]
S[7, 16] → M[{TopFace}, 1]
S[7, 28] → M[{-S[3, 37], -S[1, 7], S[1, 16], S[1, 7]}, 1 571 486 480]
S[7, 30] → M[{S[7, 39], S[7, 16]}, 3]
S[7, 31] → M[{-S[3, 36], -S[2, 25], -S[1, 36], S[7, 48], S[1, 12]}, 1 098 423 165 509]
S[7, 33] → M[{-S[3, 33], -S[2, 8], -S[1, 13], S[9, 30], S[1, 13]}, 236 364 402 983]
S[7, 34] → M[{S[30, 39], S[7, 37]}, 44 864 783 719]
S[7, 36] → M[{-S[3, 37], -S[2, 42], -S[1, 3], S[7, 33], S[1, 3]}, 236 373 334 160]
S[7, 37] → M[{-S[3, 34], -S[2, 42], -S[1, 3], S[7, 45], S[1, 3]}, 1 592 418 690]
S[7, 39] → M[{S[7, 16], S[7, 16]}, 2]
S[7, 43] → M[{-S[3, 18], -S[2, 38], -S[1, 7], S[2, 21], S[1, 7]}, 51 458 173 621]
S[7, 45] → M[{-S[3, 33], -S[2, 38], -S[1, 3], S[7, 46], S[1, 3]}, 1 586 456 543]
S[7, 46] → M[{-S[3, 39], -S[2, 19], -S[1, 28], S[7, 28], S[1, 7]}, 1 580 451 495]
S[7, 48] → M[{-S[3, 46], -S[2, 21], -S[1, 9], S[1, 7], S[1, 12]}, 541 800 208 034]
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S[8, 14] → M[{-S[1, 7], S[24, 25], S[1, 7]}, 171 564 174 758]
S[8, 19] → M[{-S[3, 37], -S[2, 27], -S[1, 7], S[21, 25], S[1, 7]}, 7 011 205 843]
S[8, 21] → M[{-S[7, 12], -S[3, 48], -S[2, 44], -S[1, 36], S[7, 36], S[1, 7]}, 243 429 270 328]
S[8, 22] → M[{-S[7, 34], -S[6, 44], -S[4, 40], -S[3, 34], -S[2, 27], S[4, 11], S[2, 6]}, 192 16
S[8, 24] → M[{-S[3, 33], -S[1, 7], S[24, 38], S[1, 7]}, 12 976 466 236 803]
S[8, 25] → M[{-S[7, 37], -S[3, 34], -S[2, 19], -S[1, 13], S[3, 13], S[1, 3]}, 3 393 675 957]
S[8, 27] → M[{-S[3, 34], -S[1, 3], S[8, 38], S[1, 3]}, 343 462 338]
S[8, 29] → M[{-S[3, 15], -S[2, 19], -S[1, 7], S[19, 51], S[1, 7]}, 3 231 023 217 992]
S[8, 32] → M[{-S[3, 39], -S[2, 27], -S[1, 3], S[8, 35], S[1, 3]}, 84 220 170 377 495]
S[8, 35] → M[{-S[3, 15], -S[2, 27], -S[1, 7], S[24, 27], S[1, 7]}, 84 220 164 406 193]
S[8, 38] → M[{-S[2, 40], -S[1, 7], S[24, 35], S[1, 7]}, 337 500 262]
S[8, 40] → M[{-S[2, 27], -S[1, 54], S[2, 8], S[1, 3]}, 173 056 376]
S[8, 42] → M[{-S[3, 46], -S[1, 28], S[8, 44], S[1, 45]}, 66 310 702 038]
S[8, 44] → M[{-S[3, 34], -S[2, 19], -S[1, 7], S[21, 27], S[1, 7]}, 66 310 701 526]
S[8, 47] → M[{-S[2, 8], S[29, 44], S[2, 8]}, 17 514 175 509 023]
S[8, 49] → M[{-S[3, 33], -S[2, 27], -S[1, 16], S[8, 32], S[1, 31]}, 84 221 720 991 751]
S[8, 51] → M[{-S[7, 13], -S[3, 46], -S[2, 27], -S[1, 7], S[8, 24], S[1, 7]}, 12 976 675 084 141]
S[9, 15] → M[{-S[8, 19], -S[3, 31], -S[2, 38], -S[1, 7], S[19, 21], S[1, 7]}, 21 472 639 202]
S[9, 16] → M[{-S[3, 9], -S[2, 38], -S[1, 3], S[9, 48], S[1, 3]}, 65 604 252]
S[9, 28] → M[{-S[1, 3], S[9, 39], S[1, 3]}, 1 541 683 115]
S[9, 30] → M[{-S[3, 39], -S[2, 21], S[9, 43], S[2, 21]}, 235 797 502 277]
S[9, 31] → M[{-S[1, 7], S[33, 48], S[1, 7]}, 1 529 774 967]
S[9, 33] → M[{-S[7, 16], -S[3, 46], -S[1, 30], S[1, 7], S[1, 3]}, 11 932 938]
S[9, 34] → M[{-S[3, 18], -S[2, 19], -S[1, 3], S[9, 37], S[1, 3]}, 29 803 364]
S[9, 36] → M[{-S[8, 25], -S[7, 28], -S[3, 13], -S[2, 19], -S[1, 3], S[7, 34], S[1, 3]}, 51 434 2
S[9, 37] → M[{-S[3, 45], -S[1, 30], S[1, 9], S[1, 3]}, 23 848 990]
S[9, 39] → M[{-S[1, 3], S[9, 31], S[1, 3]}, 1 535 729 041]
S[9, 43] → M[{-S[8, 42], -S[1, 15], S[8, 42], S[1, 9]}, 132 917 030 414]
S[9, 45] → M[{-S[8, 21], -S[3, 54], -S[2, 38], S[8, 21], S[2, 40]}, 486 858 660 585]
S[9, 46] → M[{-S[3, 18], -S[1, 3], S[9, 34], S[1, 3]}, 35 757 554]
S[9, 48] → M[{-S[3, 46], -S[1, 3], S[9, 46], S[1, 3]}, 41 712 135]
S[19, 21] → M[{-S[3, 54], -S[1, 3], S[21, 42], S[1, 3]}, 14 443 470 501]
S[19, 22] → M[{-S[9, 36], -S[3, 28], -S[2, 19], -S[1, 12], S[2, 35], S[1, 12]}, 41 305 265 968 60
S[19, 24] → M[{-S[3, 37], -S[2, 19], -S[1, 3], S[21, 35], S[1, 3]}, 1 076 897 234 907]
S[19, 25] → M[{-S[9, 16], -S[3, 30], -S[2, 40], -S[1, 3], S[9, 16], S[1, 3]}, 143 201 902]
S[19, 27] → M[{-S[3, 18], -S[1, 10], S[28, 46], S[1, 10]}, 1566]
S[19, 29] → M[{-S[1, 3], S[29, 42], S[1, 3]}, 9 693 478 809 322]
S[19, 32] → M[{-S[1, 28], S[32, 44], S[1, 28]}, 17 341 876 789 811]
S[19, 35] → M[{-S[3, 31], -S[2, 38], -S[1, 3], S[27, 32], S[1, 3]}, 17 284 664 263 771]
S[19, 38] → M[{-S[3, 54], -S[1, 10], S[3, 31], S[1, 10]}, 16 928]
S[19, 40] → M[{-S[3, 48], -S[2, 19], -S[1, 10], S[3, 43], S[1, 10]}, 17 183]
S[19, 42] → M[{-S[3, 46], -S[2, 19], -S[1, 10], S[3, 46], S[1, 10]}, 1232]
S[19, 44] → M[{-S[1, 10], S[3, 18], S[1, 10]}, 150]
S[19, 47] → M[{-S[3, 28], -S[1, 3], S[19, 32], S[1, 3]}, 17 341 882 760 338]
S[19, 49] → M[{-S[1, 7], S[2, 53], S[1, 7]}, 57 178 123 796]
S[19, 51] → M[{-S[3, 18], -S[1, 3], S[24, 40], S[1, 3]}, 3 230 733 537 405]
S[21, 22] → M[{-S[2, 53], -S[1, 7], S[8, 14], S[1, 7]}, 228 742 298 554]
S[21, 24] → M[{-S[19, 35], -S[3, 18], -S[2, 19], -S[1, 12], S[21, 47], S[1, 12]}, 39 010 257 513
S[21, 25] → M[{-S[19, 25], -S[3, 31], -S[2, 38], -S[1, 3], S[21, 38], S[1, 3]}, 6 990 333 578]
S[21, 27] → M[{-S[19, 38], -S[1, 31], S[2, 25], S[1, 45]}, 66 292 798 118]
S[21, 29] → M[{-S[19, 38], -S[3, 46], -S[1, 3], S[21, 32], S[1, 3]}, 1 076 909 177 673]
S[21, 32] → M[{-S[19, 40], -S[1, 3], S[19, 24], S[1, 3]}, 1 076 903 206 164]
S[21, 35] → M[{-S[19, 25], -S[9, 30], -S[2, 32], -S[1, 3], S[27, 29], S[1, 3]}, 1 076 888 303 617
S[21, 38] → M[{-S[19, 44], -S[3, 46], -S[2, 38], -S[1, 3], S[21, 44], S[1, 3]}, 6 841 109 966]
S[21, 40] → M[{-S[19, 21], -S[9, 48], -S[8, 38], -S[7, 30], -S[3, 7], -S[2, 27], -S[1, 3], S[7
S[21, 42] → M[{-S[19, 44], -S[8, 38], -S[7, 12], -S[3, 31], -S[2, 40], -S[1, 3], S[7, 12], S[1
S[21, 44] → M[{-S[19, 40], -S[9, 48], -S[8, 25], -S[3, 54], -S[2, 40], -S[1, 3], S[8, 25], S[1
S[21, 47] → M[{-S[19, 51], -S[3, 30], -S[1, 7], S[8, 47], S[1, 7]}, 20 744 932 912 223]
S[21, 51] → M[{-S[19, 47], -S[9, 34], -S[8, 21], -S[7, 31], -S[6, 17], -S[3, 7], -S[2, 44], -S
S[24, 25] → M[{-S[19, 49], -S[1, 7], S[32, 38], S[1, 7]}, 171 546 279 536]
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S[24, 27] → M[{-S[21, 27], -S[19, 27], -S[9, 39], -S[7, 15], -S[4, 14], -S[2, 38], -S[1, 43],
S[24, 29] → M[{-S[3, 18], -S[2, 38], -S[1, 3], S[19, 25], S[1, 3]}, 149 207 052]
S[24, 32] → M[{-S[19, 40], -S[9, 37], -S[3, 43], -S[2, 44], -S[1, 43], S[1, 15], S[1, 3]}, 301
S[24, 35] → M[{-S[19, 38], -S[3, 46], -S[2, 42], -S[1, 3], S[24, 47], S[1, 3]}, 319 536 289]
S[24, 38] → M[{-S[19, 42], -S[3, 54], -S[2, 38], -S[1, 3], S[24, 44], S[1, 3]}, 12 976 448 341 56
S[24, 40] → M[{-S[19, 38], -S[2, 40], -S[1, 3], S[24, 42], S[1, 3]}, 3 230 727 583 215]
S[24, 42] → M[{-S[21, 32], -S[19, 24], -S[3, 30], -S[2, 42], -S[1, 3], S[21, 29], S[1, 3]}, 3 2
S[24, 44] → M[{-S[19, 51], -S[9, 15], -S[1, 7], S[32, 42], S[1, 7]}, 12 976 442 335 083]
S[24, 47] → M[{-S[19, 40], -S[3, 30], -S[2, 42], -S[1, 3], S[24, 32], S[1, 3]}, 313 564 709]
S[24, 51] → M[{-S[19, 27], -S[8, 38], -S[3, 34], -S[2, 27], -S[1, 3], S[8, 27], S[1, 3]}, 686 9
S[27, 29] → M[{-S[24, 32], -S[19, 40], -S[9, 34], -S[3, 9], -S[2, 22], -S[1, 46], S[2, 32], S[
S[27, 32] → M[{-S[21, 35], -S[19, 42], -S[9, 16], -S[8, 29], -S[3, 34], -S[2, 40], -S[1, 3], S
S[27, 35] → M[{-S[19, 49], -S[1, 3], S[19, 35], S[1, 3]}, 17 341 848 341 641]
S[27, 38] → M[{-S[19, 42], -S[3, 54], -S[1, 18], S[29, 47], S[1, 18]}, 155 926]
S[27, 40] → M[{-S[3, 48], -S[1, 10], S[2, 38], S[1, 10]}, 50 997]
S[27, 42] → M[{-S[3, 18], -S[1, 10], S[3, 54], S[1, 10]}, 368]
S[27, 44] → M[{-S[3, 34], -S[2, 44], -S[1, 10], S[3, 34], S[1, 10]}, 16 440]
S[27, 47] → M[{-S[24, 25], -S[21, 32], -S[19, 21], -S[3, 33], -S[1, 10], S[21, 24], S[1, 10]},
S[27, 51] → M[{-S[2, 53], -S[1, 3], S[19, 49], S[1, 3]}, 114 344 306 444]
S[28, 30] → M[{-S[27, 40], -S[19, 40], -S[3, 46], -S[2, 38], -S[1, 18], S[27, 38], S[1, 18]}, 2
S[28, 31] → M[{FrontFace}, 1]
S[28, 33] → M[{-S[19, 38], -S[3, 31], -S[2, 42], -S[1, 10], S[3, 39], S[1, 10]}, 50 926]
S[28, 34] → M[{-S[27, 42], -S[19, 38], -S[1, 28], S[32, 35], S[1, 28]}, 2 977 030]
S[28, 36] → M[{-S[27, 40], -S[3, 28], -S[1, 10], S[2, 40], S[1, 10]}, 136 235]
S[28, 37] → M[{-S[27, 44], -S[19, 42], -S[3, 33], -S[1, 10], S[27, 40], S[1, 10]}, 68 717]
S[28, 39] → M[{-S[27, 42], -S[19, 38], -S[2, 19], -S[1, 10], S[32, 47], S[1, 10]}, 481 606]
S[28, 43] → M[{-S[27, 42], -S[19, 42], -S[1, 10], S[19, 27], S[1, 10]}, 3200]
S[28, 45] → M[{-S[27, 44], -S[19, 42], -S[3, 28], -S[2, 19], -S[1, 10], S[28, 39], S[1, 10]}, 5
S[28, 46] → M[{-S[19, 44], -S[1, 10], S[19, 42], S[1, 10]}, 1416]
S[28, 48] → M[{-S[27, 44], -S[3, 33], -S[1, 18], S[28, 45], S[1, 18]}, 532 405]
S[29, 32] → M[{-S[28, 37], -S[27, 44], -S[19, 38], -S[3, 28], -S[2, 38], -S[1, 10], S[30, 37],
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S[29, 38] → M[{-S[28, 33], -S[24, 47], -S[21, 38], -S[19, 21], -S[9, 48], -S[8, 25], -S[7, 30]
S[29, 40] → M[{-S[28, 37], -S[27, 38], -S[19, 40], -S[3, 39], -S[2, 40], -S[1, 3], S[29, 38],
S[29, 42] → M[{-S[28, 46], -S[24, 51], -S[19, 51], -S[8, 29], -S[3, 18], -S[1, 3], S[8, 29], S
S[29, 44] → M[{-S[28, 45], -S[27, 42], -S[24, 25], -S[19, 38], -S[9, 33], -S[1, 13], S[19, 47]
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S[30, 31] → M[{-S[28, 46], -S[1, 31], S[33, 34], S[1, 31]}, 7760]
S[30, 33] → M[{-S[28, 46], -S[27, 42], -S[19, 42], -S[2, 19], -S[1, 3], S[30, 36], S[1, 3]}, 25
S[30, 34] → M[{-S[27, 42], -S[19, 42], -S[2, 19], -S[1, 3], S[30, 48], S[1, 3]}, 240 398 413]
S[30, 36] → M[{-S[29, 35], -S[28, 33], -S[27, 42], -S[3, 46], -S[2, 42], -S[1, 3], S[30, 34],
S[30, 37] → M[{-S[29, 47], -S[28, 39], -S[19, 44], -S[3, 39], -S[1, 18], S[28, 48], S[1, 18]},
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S[32, 38] → M[{-S[1, 3], S[27, 51], S[1, 3]}, 114 350 260 518]
S[32, 42] → M[{-S[30, 33], -S[29, 38], -S[28, 36], -S[24, 35], -S[19, 40], -S[9, 34], -S[3, 16
S[32, 44] → M[{-S[30, 31], -S[29, 35], -S[27, 42], -S[19, 40], -S[3, 39], -S[2, 38], -S[1, 33]
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