Non Commutative Gaussian

Elimination - Program 1

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Amended from a similar notebook by Dror Bar-Natan and Itai Bar-Natan. The original version is at http://www.math.toronto.edu/~drorbn/Misc/SchreierSimsRubik/.

Pensieve Header: NCGE Program 1 - keeping track of lengths of tricks; the results are sad.

The Cube

The Generating Permutations

```mathematica
n = 54; $RecursionLimit = 2^16;
Generators = {
M[{{18, 27, 36, 4, 5, 6, 7, 8, 9, 3, 11, 12, 13, 14, 15, 16, 17,
   45, 2, 20, 21, 22, 23, 24, 25, 26, 44, 1, 29, 30, 31, 32, 33, 34, 35, 43,
   37, 38, 39, 40, 41, 42, 10, 19, 28, 52, 49, 46, 53, 50, 47, 54, 51, 48},
   {BottomFace}, 1],
M[{{1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 9, 15, 24, 33, 39, 17,
   18, 19, 20, 8, 14, 23, 32, 26, 27, 28, 29, 7, 13, 22, 31, 37, 35, 36,
   12, 21, 30, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54},
   {TopFace}, 1],
M[{{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17,
   18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 31, 32, 33, 34, 35, 36, 48, 47, 46,
   39, 42, 45, 38, 41, 44, 37, 40, 43, 30, 29, 28, 49, 50, 51, 52, 53, 54},
   {FrontFace}, 1],
M[{{3, 6, 9, 2, 5, 8, 1, 4, 7, 54, 53, 52, 10, 11, 12, 13, 14,
   15, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36,
   37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 18, 17, 16},
   {BackFace}, 1],
M[{{13, 2, 3, 22, 5, 6, 31, 8, 9, 12, 21, 30, 37, 14, 15, 16,
   17, 18, 11, 20, 29, 40, 23, 24, 25, 26, 27, 10, 19, 28, 43, 32, 33, 34, 35,
   36, 46, 38, 39, 49, 41, 42, 52, 44, 45, 1, 47, 48, 4, 50, 51, 7, 53, 54},
   {LeftFace}, 1],
M[{{1, 2, 48, 4, 5, 51, 7, 8, 54, 10, 11, 12, 13, 14, 3, 18, 27,
   36, 19, 20, 21, 22, 23, 6, 17, 26, 35, 28, 29, 30, 31, 32, 9, 16, 25, 34,
   37, 38, 15, 40, 41, 42, 43, 44, 33, 46, 47, 39, 49, 50, 52, 53, 45},
   {RightFace}, 1]};
```
**Program 1**

```
Clear[s, M];
M /: M[a1_, {w1___}, m1_] ** M[a2_, {w2___}, m2_] := M[a1[[a2]], {w1, w2}, m1 + m2];
M /: Inverse[M[a_, {w_}, m_]] := M[Ordering[a], -Reverse[{w}], m];
Feed[M[Range[n], ___]] := Null;
Feed[M[a_, {w___}, m_]] := Module[{i, j, k, l},
  For[i = 1, a[[i]] = i, ++i; j = a[[i]];
    If[Head[s[i, j]] == M,
      Feed[ReplacePart[Inverse[s[i, j]] ** M[a, {w}, m], {S[i, j], w}, 2]],
      s[i, j] = M[a, {w}, m];
    Do[
      If[Head[s[k, l]] == M,
        Feed[ReplacePart[s[i, j] ** S[k, l], {S[i, j], S[k, l]}, 2]],
        Feed[ReplacePart[s[k, l] ** S[i, j], {S[k, l], S[i, j]}, 2]]
      ],
      {k, n}, {l, n}
    ];
  ];
  Images[i_] := Prepend[Select[Range[n], Head[s[i, n]] == M &], i]
]
Timing[
  (Feed[#]; Product[Length[Images[i]], {i, n}]) & /@ Generators
]
{103.865, {4, 16, 159993501696000, 21119142223872000, 43252003274489856000, 4325200327448985600}}
```

The Order of the Group

```
Timing[
  (Feed[#]; Product[Length[Images[i]], {i, n}]) & /@ Generators
]
{103.865, {4, 16, 159993501696000, 21119142223872000, 43252003274489856000, 43252003274489856000}}
```

It is lovely to note that the number computed right above, the order of the Rubik's cube group according to our computer, agrees with the number that appears in the literature, for example, in Wikipedia. Note also that according to our computation the last generator of the group was actually not necessary.

The Worst Case Scenario

```
Sum[Max[Last[s[i, n]] & /@ Images[i]], {i, n}]
1059923953939810
```

For the Patient: A Recipe for Solving the Cube

```
Reap[
  Do[If[Head[s[i, j]] == M, Sow[S[i, j] -> Rest[s[i, j]]]], {i, n - 1}, {j, i + 1, n}]
][[2, 1]] // ColumnForm
S[1, 3] -> M[({S[3, 37], S[1, 31]}, 2977037]
S[1, 7] -> M[({S[3, 7], S[1, 3]}, 8947611]
S[1, 9] -> M[({S[3, 9], S[1, 3]}, 20864120]
```

http://drorbn.net/AcademicPensieve/2009-07/#MathematicaNotebooks