Non Commutative Gaussian Elimination - Program 1

By Dror Bar-Natan

Amended from a similar notebook by Dror Bar-Natan and Itai Bar-Natan. The original version is at http://www.math.toronto.edu/~drorbn/Misc/SchreierSimsRubik/.

Pensieve Header: NCGE Program 1 - keeping track of lengths of tricks; the results are sad.

The Cube


http://drorbn.net/AcademicPensieve/2009-07/#MathematicaNotebooks
IGA4K = eKFE'^M800 =0000 AFeRiFA ifPcF9ZIF - d81 '081 L66''=6mSHFeU4k0''
\Large /F /HdA_Hge] /Ifid;
\Large T = KQ5 \sim ^&g1bIF5TLFQU

< NCGEProgram1.nb

\Large http://drorbn.net/AcademicPensieve/2009-07/#MathematicaNotebooks

2017-02-11 19:41:06
The Generating Permutations

http://drorbn.net/AcademicPensieve/2009-07/#MathematicaNotebooks
n = 54; $RecursionLimit = 2^16;
Generators = {
    M[{18, 27, 36, 4, 5, 6, 7, 8, 9, 3, 11, 12, 13, 14, 15, 16, 17,
       45, 2, 20, 21, 22, 23, 24, 25, 26, 44, 1, 29, 30, 31, 32, 33, 34, 35, 43,
       37, 38, 39, 40, 41, 42, 10, 19, 28, 52, 49, 46, 53, 50, 47, 54, 51, 48},
    {BottomFace}, 1],
    M[{1, 2, 3, 4, 5, 6, 16, 25, 34, 10, 11, 9, 15, 24, 33, 39, 17,
       18, 19, 20, 8, 14, 23, 32, 38, 26, 27, 28, 29, 7, 13, 22, 31, 37, 35, 36,
       12, 21, 30, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54},
    {TopFace}, 1],
    M[{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17,
       18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 31, 32, 33, 34, 35, 36, 48, 47, 46,
       39, 42, 45, 38, 41, 44, 37, 40, 43, 30, 29, 28, 49, 50, 51, 52, 53, 54},
    {FrontFace}, 1],
    M[{3, 6, 9, 2, 5, 8, 1, 4, 7, 54, 53, 52, 10, 11, 12, 13, 14,
       15, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36,
       37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 18, 17, 16},
    {BackFace}, 1],
    M[{13, 2, 3, 22, 5, 6, 31, 8, 9, 12, 21, 30, 37, 14, 15, 16,
       17, 18, 11, 29, 40, 23, 24, 25, 26, 27, 10, 19, 28, 43, 32, 33, 34, 35,
       36, 46, 38, 39, 40, 41, 42, 52, 44, 45, 1, 47, 48, 4, 50, 51, 7, 53, 54},
    {LeftFace}, 1],
    M[{1, 2, 48, 4, 5, 51, 7, 8, 54, 10, 11, 12, 13, 14, 3, 18, 27,
       36, 19, 20, 21, 22, 23, 6, 17, 26, 35, 28, 29, 30, 31, 32, 9, 16, 25, 34,
       37, 38, 15, 40, 41, 24, 43, 44, 33, 46, 47, 39, 49, 50, 42, 52, 53, 45},
    {RightFace}, 1]};
Program 1

Clear[s, M];
M :> M[a1_, {w1___}, m1_] ** M[a2_, {w2___}, m2_] := M[a1[[a2]], {w1, w2}, m1 + m2];
M :> Inverse[M[a_, w_, m_]] := M[Ordering[a], -Reverse[w], m];
Feed[M[Range[n], ___]] := Null;
Feed[M[a_, {w___}, m_]] := Module[{i, j, k, l},
  For[i = 1, a[[i]] == i, ++i; j = a[[i]];
  If[Head[s[i, j]] === M,
    Feed[ReplacePart[Inverse[s[i, j]] ** M[a, {w}, m], {-S[i, j], w}, 2]],
    s[i, j] = M[a, {w}, m];
  Do[
    If[Head[s[k, l]] === M,
      Feed[ReplacePart[s[i, j] ** s[k, l], {S[i, j], S[k, l]}, 2]];
      Feed[ReplacePart[s[k, l] ** s[i, j], {S[k, l], S[i, j]}, 2]]
    ],
    {k, n}, {l, n}
  ]
],
Images[i_] := Prepend[Select[Range[n], Head[s[i, #]] === M &], i]

The Order of the Group

Timing[
  (Feed[n]; Product[Length[Images[i]], {i, n}] & /@ Generators
]
{103.865, {4, 16, 159993501696000,
  2111942223872000, 43252003274489856000, 43252003274489856000}}

It is lovely to note that the number computed right above, the order of the Rubik's cube group according to our computer, agrees with the number that appears in the literature, for example, in Wikipedia. Note also that according to our computation the last generator of the group was actually not necessary.

The Worst Case Scenario

Sum[Max[Last[s[i, #]] & /@ Images[i]], {i, n}]
1059923953939810

For the Patient: A Recipe for Solving the Cube

Reap[
  Do[If[Head[s[i, j]] === M, Sow[S[i, j] -> Rest[s[i, j]]]], {i, n - 1}, {j, i + 1, n}]
][[2, 1]] // ColumnForm
S[1, 3] -> M[[S[3, 37], S[1, 31]], 2977037]
S[1, 7] -> M[[S[3, 7], S[1, 3]], 8947611]
S[1, 9] -> M[[S[3, 9], S[1, 3]], 20864120]