Dror Bar-Natan: Misc: Academic Pensieve: 2009-07/
MMRForKAL-090708-Annotated.pdf

From "On the Melvin-Morton-Rozansky Conjecture" by Bar-Natan and Garoufalidis.

6.1. **Immanants and the Conway polynomial.** Theorem 3 and proposition 4.2 show (in particular) that both the map $D \mapsto \det \mathrm{IM}(D)$ and the map $D \mapsto \operatorname{per} \mathrm{IM}(D)$ are weight systems. It is tempting to look for common generalizations of these two weight systems. In this section, which may be of some independent interest, we sketch just such a generalization. The basic idea is that just where the character of the alternating representation of the symmetric group S_m is used in the definition of det and the character of the trivial representation is used in the definition of per, one can put the character of an arbitrary representation of S_m:

Definition 6.1. Let $[\sigma]$ denote the conjugacy class of a permutation σ. Let ZS_m be the free $\mathbf{Z}$-module generated by the conjugacy classes of S_m. Let $ZS_\star$ be the graded $\mathbf{Z}$-module whose degree m piece is ZS_m. The natural embedding $\iota : S_m \times S_n \to S_{m+n}$ makes $ZS_\star$ an algebra by setting $[\sigma][\tau] = [\iota(\sigma,\tau)]$. Identifying $ZS_\star$ with its dual by declaring each individual conjugacy class $[\sigma]$ to be of unit norm, the product on $ZS_\star$ becomes a co-product on $ZS_\star^\star = ZS_\star$.

Exercise 6.2. Verify that with the above product and co-product $ZS_\star$ becomes a graded commutative and co-commutative Hopf algebra, and that the primitive elements of $ZS_\star$ are exactly the classes of cyclic permutations (and thus $ZS_\star$ has exactly one generator in each degree).

Definition 6.3. (Compare with [Lit]) Let M be an $m \times m$ matrix. The *universal immanant* $\operatorname{imm} M$ of M is defined by

$$\operatorname{imm} M = \sum_{\sigma \in S_m} [\sigma] \prod_{i=1}^{m} M_{i\sigma i} \in ZS_m.$$

(Exactly the same as the definition of $\det M$, only with $[\sigma]$ replacing $(-1)^\sigma$).

Composing the universal immanant with characters of arbitrary representations of S_m, one gets specific complex valued "immanants". Taking the representation to be the alternating representation, one gets $\det M$. Taking it to be the trivial representation, one gets $\operatorname{per} M$. Much is known about many other immanants; see e.g. [GJ, St1, St2].

In our context, we will be interested in the universal immanant of the intersection matrix of a chord diagram. By abuse of notation, we will write $\operatorname{imm} D$ for $\operatorname{imm} \mathrm{IM}(D)$.

Theorem 5. (1) *The map* imm : $\{$*chord diagrams*$\} \to ZS_\star$ *descends to a well defined map* $\operatorname{imm} : \mathcal{A}^\nabla \to ZS_\star$.
(2) *The thus defined* $\operatorname{imm} : \mathcal{A}^\nabla \to ZS_\star$ *is a morphism of Hopf algebras.*
(3) *The image of the adjoint map* $\operatorname{imm}^\star : ZS_\star^\star = ZS_\star \to \mathcal{A}^{\nabla\star} = \mathcal{W}$ *is the subalgebra of* $\mathcal{W}$ *generated by the weight systems of the coefficients of the Conway polynomial.*

Proof. (sketch) Let L_m be the degree m piece of $\log W_C$, and let $C_m \in S_m$ be a cyclic permutation. Re-interpreted in our new language, proposition 3.13 is simply the statement $\operatorname{imm}^\star[C_m] = -L_m$ and equation (14) becomes the multiplicativity of $\operatorname{imm}^\star$. It follows that the image of $\operatorname{imm}^\star$ is equal to the subalgebra of the algebra of functionals on chord diagrams generated by the L_m's. As L_m is known to be a weight system and the product of two weight systems is again a weight system, it follows that the image of $\operatorname{imm}^\star$ is in $\mathcal{W}$ and thus imm descends to $\mathcal{A}^\nabla$. Finally notice that the algebra generated by the L_m's is equal to the algebra generated by the weight systems of the coefficients of the Conway polynomial. □

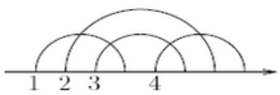
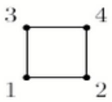
It is easy to check (or deduce from theorem 5) that $\mathrm{imm}^\star[\sigma] = 0$ if σ has a cycle of an odd length. By evaluating $\mathrm{imm}^\star[\sigma]$ on chord diagrams whose intersection graph is a union of polygons of an even number of sides, one can see that $\mathrm{imm}^\star$ restricted to permutations with no cycles of odd length is injective.

Exercise 6.4. Check that if $\mathrm{IM}(D)$ is replaced by $\mathrm{IM}(D)+\lambda I$ for any non-zero constant λ and $\mathcal{A}^\nabla$ and $\mathcal{W}$ are replaced by $\mathcal{A}$ and $\mathcal{A}^\star$ in the statement of theorem 5, the theorem remains valid, with the unique element of $\mathcal{G}_\infty\mathcal{A}^\star$ adjoined to the generators of the image of $\mathrm{imm}^\star$.

But first, the definitions:

$$\mathrm{IM}(D)_{ij} = \begin{cases} \mathrm{sign}(i-j) & \text{if chords } i \text{ and } j \text{ of } D \text{ intersect (where chords of } D \text{ are numbered from left to right),} \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.5.

$D =$ , $\mathrm{LIG}(D) =$ , $\mathrm{IM}(D) = \begin{pmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$

...and the basic theorem:

<u>Thm</u> Let W_C be the weight system of the Conway poly. Then $W_C(D) = \det(\mathrm{IM}(D))$.

<u>Proof</u> <u>Sketch</u>: Like W_{gl}, W_C satisfies the 2T relation and hence it is determined by its values on "caravans":

Thus it is enough to show that det im satisfies 2T and check the initial cond.

(There's also a nicer proof by Melvin. see exercise 3.9 in the paper)

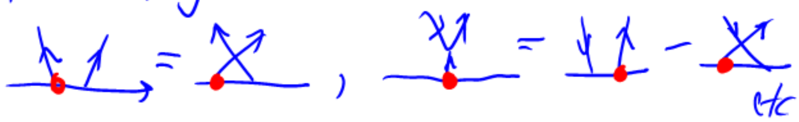
The big questions.

1. Can you extend this to w-knots & arrow diagrams?
2. Can you "globalize" this?
3. Why should we care?
   (at the moment, my only answer is, "when God gives us a toy, it means she wants us to play with it.)

Further topics.
1. No w-Conway relation for pA.
2. No 2T relation for arrow diagrams, but maybe a 3T?
3. I don't know if gl remains related to w-Alexander.

⋮

Another theological question: In the $\overrightarrow{4T}$/OC/$\overrightarrow{STU}$ relations, tails "maintain their identity":



etc.

So what?