\[ \text{Suppose } V^* = V^{-1} \text{ and } V^{-1} M V = M. \text{ Then} \]

\[ \begin{align*}
    \langle mf, f \rangle & = \langle V1, Vmf \rangle \\
    & = \langle V1, MVf \rangle = \langle 1, MF \rangle = \int MF = \int (V^*MVf) = \int(VMf) = \int MF \\
\end{align*} \]

\[ \text{What if } V^*\hat{w}(x+y) V = \hat{w}(x)\hat{w}(y)? \text{ Then} \]

\[ \begin{align*}
    \langle mw(x)w(y)f, f \rangle & = \langle 1, mw(x)w(y)f \rangle \\
    & = \langle V1, w(x+y)VMf \rangle \\
    & = \langle 1, w(x+y)MF \rangle \\
    & = \langle 1, w(x+y)MF \rangle = \int w(x+y)MF \\
\end{align*} \]

In general, suppose \( V^*xV = \rho \) and \( \rho V = V \). Set

\[ U = \sqrt{\frac{\rho}{\rho}} V \]

and get

\[ U^*U = V^*\sqrt{\frac{\rho}{\rho}} \sqrt{\frac{\rho}{\rho}} V = V^*\rho V = \frac{\rho}{\rho} \]

and also

\[ U1 = \sqrt{\frac{\rho}{\rho}} \quad \text{(assuming } V1 = 1) \]

the structural property of \( \mathcal{A}^* \)-operators: If \( f \) is invariant, then

\[ VF = (V1)f \]