Coxeter groups & Artin groups:

Fix $n > 0 \land m_{ij} \in \{2, 3, \ldots, \infty\}$ \hspace{1cm} 1 \leq i, j \leq n

$\langle s_i : s_i^2 = 1 \hspace{1cm} s_i s_j s_i \cdots s_m \underset{m_{ij}}{\overbrace{s_i s_j \cdots s_m}} = s_j s_i s_j \cdots \underset{m_{ij}}{\overbrace{s_i s_j \cdots s_m}} \rangle = W$

$A := \text{Same, but without } s_i^2 = 1 \quad \Rightarrow \quad A \rightarrow W$

Example: $m_{i,i+1} = 3$, $m_{ij} = 2$ otherwise.

gives $W = \text{symmetric group } S_{n+1}$

$A = \text{Braid group. } B_{n+1}$

If $|W| < \infty$, then $W$ acts on $\mathbb{R}^n$ by reflections in linear subspaces.

The affine case: $W$ acts on $\mathbb{R}^{n-1}$ by reflection about affine subspaces.

$W$ is "simply-laced" if $m_{ij} \in \{2, 3, \infty\}$

"right angled" if $m_{ij} \in \{2, \infty\}$

"free" if all $m_{ij} = \infty$

The Birman-Ko-Lee presentation of Artin $(A_n) = : A$

$T_{ij} = \sigma_i \sigma_{i+1} \cdots \sigma_{j-1} \sigma_{j-2} \sigma_{j-3} \cdots \sigma_{i-1}$

= \hspace{1cm} \begin{array}{c|c}
| & \\
| & \end{array} \\
\hspace{1cm} \begin{array}{c|c|c|c|c}
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\end{array} \quad i < j$
Relations \( \tau_{ik} \tau_{ij} = \tau_{jk} \tau_{ik} = \tau_{ij} \tau_{jk} \) for \( i \leq j \leq k \)

\[ \text{Thm} \quad A = \langle \tau_{ij} \rangle / \text{rels} \]

Similar presentations exists if \( |W| < \infty \) (Bassis)

\( \hat{A}_n \) (Digne) (Bassis)

\( \text{free} \) (Bassis)

Today: A uniform proof of a similar presentation in the finite, affine and right angled cases (other than the non-crystallographic finite reflection groups)

The presentation: (\( W \) is simply laced)
Let \( Q \) be the quiver of \( \bullet \ldots \bullet \)
if \( m_{ij} = 3 \) put an arrow \( \bullet \rightarrow \bullet \)
if \( m_{ij} = \infty \) put many arrows \( \bullet \rightarrow \bullet \)

A representation \( X \) of \( Q \) is "exceptional" if it is indecomposable and \( \text{Ext}^1(X, X) = 0 \).

\( E_1 \ldots E_r \) is an exceptional sequence if \( E_i \) are exceptional \& \( \text{Ext}^1(E_j, E_i) = \text{Hom}(E_j, E_i) = 0 \)

There is a presentation \( D \) of the Artin group whose generators are "exceptional quivers" and whose relations come from "mutations of exceptional sequences".