Kamnitzer: Categorical \(\mathfrak{sl}(2)\) actions and equivalences of categories

(Joint w/ Sabin Cautis & Anthony Licata)

\(X\) - smooth variety / \(\mathbb{C}\)

\(D(X) = \text{bounded derived category of coherent sheaves on } X\)

Objects: complexes of coherent sheaves (vector bundles)

Questions:
1. When is \(D(X) \cong D(Y)\)?
2. What is \(\text{Aut}(D(X))\)?

Goal: Find interesting autoequivalences of \(D(X)\)

(for certain \(X\))

\[
\text{K}_0(D(X))_\mathbb{C} = \left\{ \text{complexes with \ text{basis \ \mathcal{D}_{\mathbb{C}}} \right\} \quad \text{whenever}
\begin{align*}
[A] & \in \mathbb{H}[E] \\
\exists & 0 \to \mathcal{E} \to \mathcal{A} \to \mathcal{C} \to 0
\end{align*}
\]

(sometimes \(K_0 \cong H_0\))

Motivations:
* Purely alg. geometry
* Homological mirror symmetry \(D(X) \cong \text{Fuk}(Y)\)
* Can make braid group actions on \(D(X)\)
  \(\to\) get homological knot invariants
* Can lift automorphisms of \(H(X)\) to autoequivalences of \(D(X)\)

**Spherical twist** (Seidel & Thomas)

\(X, Y\) smooth varieties

A functor \(F: D(X) \to D(Y)\) is "spherical"

if
(i) \(F^*_R = F^*_L \otimes [2]\)
(ii) \(F \cdot F = I \otimes I[2]\)

In this case \(T(F)(A) = \text{Conc}(F^*_R(A) \to A)\)
In this case \( T_F(A) = \text{Conc}(F_{\mathcal{R}}(A) \to A) \) for \( A \in \text{D}(Y) \).

**Theorem (Sud'ko-Thomason)*** \( T_F \) is an equivalence of categories.

**Example**

If \( X = \text{Gr} \), \( \text{D}(X) = \text{D}(\text{Vect}) \), \( F \) is determined by \( E \in \text{D}(Y) \), \( F(V) = V \otimes E \).

\[
\begin{align*}
F_{\mathcal{R}}(A) &= \text{Ext}(E, A) \\
T_F(A) &= \text{Conc}(E \otimes \text{Ext}(E, A) \to A)
\end{align*}
\]

on \( K_0(\text{D}(Y)) \), this is

\[
[A] \to \dim \text{Ext}(E, A) \cdot [E] = [A] \cdot \langle E, A \rangle [E] \quad \text{reflection on } E
\]

Last at 10:28 mins.