Possible definitions:
1. Axiomatic definition.
2. Splitting Principle I, II.
3. $H^*(BO(n), \mathbb{Z}/2)$
4. Obstruction Theory I, II
5. Cohomology operations. (Steenrod Square)
6. An explicit construction of a Čech cocycle (McLaughlin)

$X$: topological space (usually paracompact or even CW)

Bundles: $\xi: E \rightarrow X$ a varying family of vector spaces over a base $X$.

(every point has a nbhd over which $\xi$ is trivial)

Examples 1. The M"obius band.
2. The Canonical/tautological line bundle over $\mathbb{RP}^n$:
   $LRP^n \rightarrow \mathbb{RP}^n$
3. Likewise for the Grassmannian $Gr_k \mathbb{R}^n$:
   $UGr_k \mathbb{R}^n \rightarrow \mathbb{R}P^k \times \mathbb{R}^k : x \in V$

4. Tangent bundles, bundles of forms, etc.

Operations 1. Pull back via $f: Y \rightarrow X$

(Example: The pull back via $x \rightarrow x^2$ of the M"obius band is the trivial)
2. Whitney Sum: The fiber over any point is the sum of the fibers at the constituents.

Examples:
1. $\text{M"ob} \oplus \text{M"ob} = \text{Trivial}$
2. $TM \oplus NM = \text{Trivial}$
3. $TS^n \oplus NS^n = \text{Trivial}$
4. $T\mathbb{RP}^n \oplus \text{Trivial line} = (\mathbb{RP}^n)^\oplus_{n+1}$

Proof of 4:

Axioms for Steifel-Whitney:

1. $\tilde{w}: E \rightarrow W_i(E) \subseteq H^i(X, \mathbb{Z}/2)$
2. $W_0(E) = 1 \in H^0$ and $W_i(E) = 0$ if $i > \dim E$
3. Naturality with respect to pullbacks.
4. The Whitney product formula:
   \[ W(\tilde{w}_1 \oplus \tilde{w}_2) = W(\tilde{w}_1) \cdot W(\tilde{w}_2) \]
   where $W = \bigwedge w_i$
5. $W_1(\mathbb{RP}^n) \neq 0$

Corollaries:
1. $W(\mathbb{RP}^n) = 1 + (x)$ (a generator of $\tilde{w}(\mathbb{RP}^n)$).

   Since $T\mathbb{RP}^n \oplus \text{Trivial} = (\mathbb{RP}^n)^\oplus_{n+1}$,
   \[ W(T\mathbb{RP}^n) = (1 + x)^{n+1} \]

2. If $M \rightarrow \mathbb{R}^n$ is an immersion,
   \[ W(TM) \cup W(NM) = 1 \]
(This is an obstruction to the existence of immersions)

The splitting principle: Given $E_X$, look for $f: Y \to X$ s.t. $\text{a. } f^*(E) = \text{a direct sum}$
$\text{b. } f^*: H^*(X) \to H^*(Y)$ is injective
(Clearly if we find such an $f$, we can compute SW classes)

Given $E$, let $\pi: PE \to X$ be the project/visualization of $E$.

\[ TT^*E = \text{L} \text{L} \text{E+} \text{an (n-1)-disk bundle} \]

So we can continue inductively...

Need to show that $TT^*$ is injective in cohomology.