The ax+b Group

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2:44 PM

$$G = \{ (\overset{\neq 0}{a}, b) = \underbrace{(ax+b)}_{f} \} \qquad a \to \partial f, \quad b \to f(0)$$

$$(a,b)\cdot(c,d) = (ac, bc+d)$$
$$(a,b)^{-1} = (a^{-1}, -b/a)$$

$\mathfrak{g} = \langle A, B \rangle$ where $e^{\alpha A} = (e^{\alpha}, 0)$ & $e^{\beta B} = (1, \beta)$

thus $$e^{-\alpha A} e^{\beta B} e^{\alpha A} = (e^{-\alpha}, 0)\cdot(1,\beta)\cdot(e^{\alpha},0) = (1, e^{\alpha}\beta)$$

and thus $[B,A] = B$; $[A,B] = -B$

Also note,
$$e^{-\beta B} e^{\alpha A} e^{\beta B} = (1,-\beta)\cdot(e^{\alpha},0)\cdot(1,\beta) = (e^{\alpha}, \beta - \beta e^{\alpha})$$

Exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & \{x+b\} \arrow[r] \arrow[d, phantom, "\cong"] & G = \{ax+b\} \arrow[r] & \{ax\} \arrow[r] \arrow[d, phantom, "\|"] & 1 \\ & \mathbb{R}^{+} & & \mathbb{R}^{*} & \end{tikzcd}$$

thus $G = \mathbb{R}^{+} \rtimes \mathbb{R}^{*}$

On the level of Lie algebras, this is

$$0 \to \langle B \rangle \longrightarrow \langle A, B \rangle \longrightarrow \langle A \rangle \longrightarrow 0$$

In matrix language,

$$G = \{(a,b)\} = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \right\} \qquad \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} c & 0 \\ d & 1 \end{pmatrix} = \begin{pmatrix} ac & 0 \\ bc+d & 1 \end{pmatrix}$$

$$\mathfrak{g} = \{(\alpha A + \beta B)\} = \left\{ \begin{pmatrix} \alpha & 0 \\ \beta & 0 \end{pmatrix} \right\} \quad \text{so } A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Indeed,

$$[A,B] = \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

In[1]:= `MatrixExp[t {{a, 0}, {b, 0}}]`

Out[1]//MatrixForm=

$$\begin{pmatrix} e^{a t} & 0 \\ \frac{b\left(-1+e^{a t}\right)}{a} & 1 \end{pmatrix}$$

$= -B$

The exponential map:

$$\exp(\alpha,\beta) := \exp(\alpha A + \beta B) = \left(e^{\alpha}, \beta \frac{e^{\alpha}-1}{\alpha}\right)$$

Conjugacy classes in G:

$$(a,b) \sim (c,d) \quad \text{iff} \quad a = c \quad \left(\begin{array}{l}\text{except } (1,0) \text{ is} \\ \text{not conjugate to} \\ \text{anything else}\end{array}\right)$$

The adjoint action on $\mathfrak{g}$:

$$(\alpha,\beta)^{(a,b)}: \quad \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}^{-1} \begin{pmatrix} \alpha & 0 \\ \beta & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}$$

$$\begin{pmatrix} \alpha & 0 \\ \beta & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a^{-1} & 0 \\ -b/a & 1 \end{pmatrix} \begin{pmatrix} a^{-1}\alpha & 0 \\ \frac{-b\alpha}{a}+\beta & 0 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ -b\alpha+\beta a & 0 \end{pmatrix}$$

Thus $$(\alpha,\beta)^{(a,b)} = (\alpha, \beta a - b\alpha)$$

and thus $(\alpha,\beta) \sim (\gamma,\delta)$ iff $\alpha = \gamma$, except $(0,0)$ is not conjugate to anything else.