Guth on Sobolev & Loomis-Whitney

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$f: \mathbb{R}^n \to \mathbb{R}$ Q If ∇f is small, is f close to a constant?

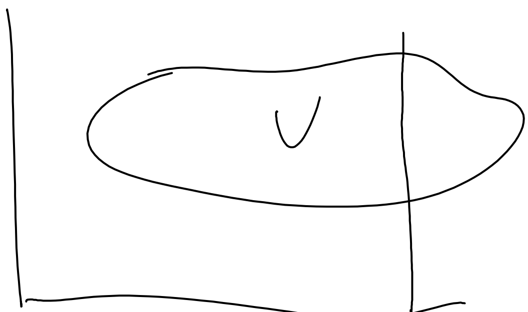
$\underline{n=1}$ $\underline{\text{Prop}}$ If $f: \mathbb{R} \to \mathbb{R}$ has $\|\nabla f\|_1 < 1$
compact support.

then $\|f\|_1 < 1$

False in $\mathbb{R}^2$ because of tall, thin functions.
Yet in $\mathbb{R}^2$,

$$m(\underbrace{[|f(x)| \geq 1]}_{U}) \leq 1$$

Pf



along every vertical line, $\int \nabla f \geq 1$
by Fubini, the length of the projection of U on the x axis is ≤ 1
$\Rightarrow$ also on the y axis
So $U \subset A \times B$ where
$m(A) \leq 1$, $m(B) \leq 1$

$\Rightarrow m(U) \leq 1$

The Sobolev inequality:
In this case, $\int |f|^2 < 10$

Prop 3 $f:\mathbb{R}^3 \to \mathbb{R}$ $\int|\nabla f| \le 1$, then

$$m(\underbrace{[|f|\ge 1]}_{U}) \le 1$$

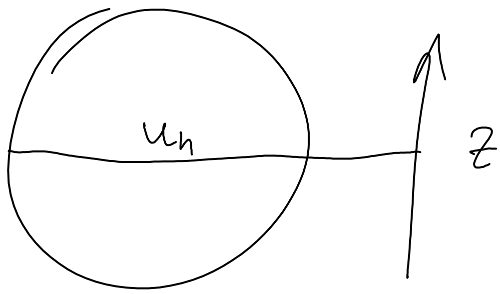
Pf The area of every projection of U on a 2D plane is ≤ 1. The rest follows from the following.

Thm (Loomis-Whitney 49') If $U \subset \mathbb{R}^3$ is open and

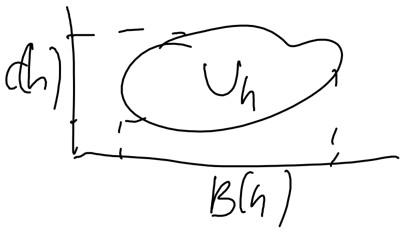
$$m(P_{xy}(U)) = A$$
$$m(P_{xz}(U)) = B \quad \text{the } m(U) \le \sqrt{ABC}$$
$$m(P_{yz}(U)) = C$$

exact for boxes.

Pf



$$U_h = U \cap [z=h]$$
$$Vol(U) = \int_{-\infty}^{\infty} dh\, Area(U_h)$$
$$Area(U_h) \le A$$
$$\le B(h)C(h)$$



$$\int B(h)dh = B$$
$$\int c(h)\,dh = C$$

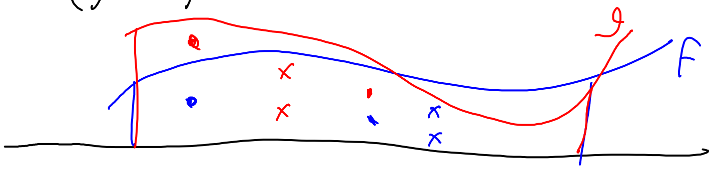
$$\Rightarrow \int B(h)^{1/2} C(h)^{1/2} dh \le \sqrt{BC} \quad \text{(Cauchy Schwartz)}$$

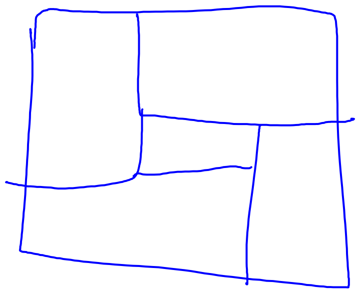
$$Area(U_h) \le A^{1/2} B(h)^{1/2} C(h)^{1/2}$$

$$\Rightarrow \int Area(U_h) \le \sqrt{ABC}$$

Cauchy ~ Schwartz:

$$\left(\int |fg|\right)^2 \leq \int |f|^2 \int |g^2|$$





$$\frac{a+b}{2} \geq \sqrt{ab}$$

$$(a+b)^2 \geq 4ab$$

$$(a+b+c)^3 \geq 27abc$$

$$(a_1b_1 + a_2b_2)^2 \leq (a_1^2 + a_2^2)(b_1^2 + b_2^2)$$

$$\|(a_1+ia_2)(b_1-ib_2)\|^2 = \|a_1+ia_2\|^2 \|b_1+ib_2\|^2$$