Vladimir Fock on Classical and Quantum Dilogarithm

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$$Li_2(x) := \sum \frac{x^i}{i^2} = -\int_0^x \log(1+y)\, d\log y \qquad \begin{pmatrix} dLi_2(x) \\ = \log(1+x) d\log x \end{pmatrix}$$

$$L(x) = Li_2(x) + \frac{\log(x)\log(x+1)}{2}$$

satisfies

$$L(a) + L(b) + L(a^{-1} + a^{-1}b)$$
$$+ L(b^{-1} + b^{-1}a) + L(a^{-1}b^{-1} + a^{-1} + b^{-1})$$
$$= -\frac{\pi^2}{2}$$

<u>Proof</u> $(\mathbb{C}^*)^2 = \{(a, b)\}$ $\quad -d\log a \wedge d\log b =: w$

$$f : \begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} b \\ \frac{b+1}{a} \end{pmatrix} \qquad f^5 = id$$

<u>Obs</u> f preserves w

$$\alpha := d^{-1} w = \log b \cdot d\log a$$

$$f^* \alpha = \alpha + d(-\log a \log b - Li_2(b))$$

$$\Rightarrow \quad \sum_{i=0}^{4} \left((f^*)^i - (f^*)^{i-1}\right) \alpha = 0$$

$\qquad ||$

can be expressed as a sum of dLi_2 ...

$\Rightarrow$ LHS is a constant ...

Branching points at $(0, -1)$

$$\phi^{\hbar}(x) = \exp -\frac{1}{4} \int \frac{e^{ipx}}{sh(\pi p)\, sh(\hbar\pi p)} \frac{dp}{p}$$

"the quantum dilogarithm"

contour around 0.

around 0.

$$\phi^\hbar(x) \xrightarrow[\hbar\to 0]{} \exp\left(\frac{Li_2(e^x)}{2\pi i \hbar}\right) \qquad |\phi^\hbar(x)| = 1 \text{ for } x \in \mathbb{R}$$

equiv to the integral expression defines $\phi^\hbar$ uniquely

$$\left\{\begin{array}{ll} \phi^\hbar(x+2\pi i\hbar) = \phi^\hbar(x)(1+qe^x) & q = e^{\pi i \hbar} \quad (*) \\ \phi^\hbar(x+2\pi i) = \phi^\hbar(x)(1+q^\vee e^{x/\hbar}) & q^\vee = e^{\pi i/\hbar} \end{array}\right.$$

$\phi^\hbar(x)$ is meromorphic & single valued.

$$\phi^\hbar(x) \underset{\hbar=1}{=} \exp\left(\frac{\pi^2}{6} - Li_2(1-e^z)\right)\Big/2\pi i$$

using residues

$$= \exp\left(Li_2(e^x) + x\log(1-e^x)\right)\Big/2\pi i$$

r, s integers.

$$\phi^{\frac{r}{s}\hbar}(rx) = \prod_m \prod_l \phi^\hbar\left(x + \frac{2\pi i}{\hbar s} m + \frac{2\pi i}{r} l\right)$$

in some range.

$$\phi^\hbar(x) = \phi^{\hbar^{-1}}(x/\hbar) \qquad \phi^\hbar(x) = \prod_{n\in\mathbb{Z}_{>0}} (1+q^{2n+1}e^x)$$

$$A^\hbar = \langle x, y : qxy = q^{-1}yx\rangle$$

$$F^\hbar : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} y^{-1} \\ x(1+qy) \end{pmatrix}$$

The $q\to 1$ limit

$\{x,y\} = xy$

$\{\log x, \log y\} = 1$

<u>Claim</u> $(F^\hbar)^5 = id.$

Factor as

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x(1+qy) \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} y^{-1} \\ x(1+qy) \end{pmatrix}$$

algebra map, also given by

"monomial" "Fourier trasform"

map given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \phi^{h}(\log y)^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \phi^{h}(\log y) = \begin{pmatrix} x(1+qy) \\ y \end{pmatrix}$$

Follows from $\quad xF(y) = F(q^{-2}y)x \; \ldots$

use $(*)$