KV for Measures

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2:08 PM

$m: \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathfrak{g}$ is $x, y \mapsto x+y$

$M: \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathfrak{g}$ is $x, y \longmapsto \log e^x e^y$

$$\begin{tikzcd} \mathfrak{g}\times\mathfrak{g} \arrow[r, shift left, "M"] \arrow[r, shift right, "m"'] \arrow[d, "\exp"] & \mathfrak{g} \arrow[d, "\exp"] \\ G\times G \arrow[r, "M"] & G \end{tikzcd}$$

<u>Claim</u> up to ν's,

$M_* = m_*$

on $G\times G$-invariants

Added Dec 9, 2008: An Abstract setup: [tentative]

($H = G\times G$, for us) $\longleftrightarrow$

$$\begin{tikzcd} H \arrow[d, "\rho"] & G \arrow[d] \\ X \arrow[r, "\alpha, \beta"] & Y \end{tikzcd}$$

$\alpha \neq \beta$ yet

$\alpha_* = \beta_*$ on H-invariants.

When does this happen?

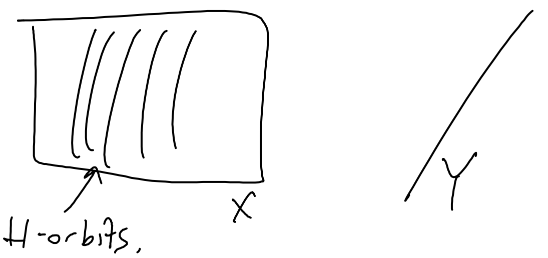
$\Updownarrow$

$$\begin{tikzcd} X\times H \arrow[r, "\rho"] \arrow[rr, bend right, "\hat{\alpha}, \hat{\beta}"'] & X \arrow[r, "\alpha, \beta"] & Y \end{tikzcd}$$

with ν the Haar measure on H,

$$\hat{\alpha}_*(\mu\times\nu) = \hat{\beta}_*(\mu\times\nu)$$

for all $\mu \in \mathcal{M}(X)$



<u>A neighboring question:</u> (the restriction of the above question to a single orbit)

Suppose (X, ν) is a measure space and $\alpha, \beta: X \longrightarrow Y$ are transformations such that $\alpha_*\nu = \beta_*\nu$. Is it reasonable to insist that the equality $\alpha_*\nu = \beta_*\nu$ be derived from the existence of a commutative triangle

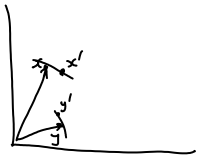
$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "\alpha"'] & & X \arrow[dl, "\beta"] \\ & Y & \end{tikzcd}$$

in which T is measure preserving? (i.e., $T_* \nu = \nu$)

Aside: if $G \circlearrowright X$ then $G \times G \circlearrowright X \times X$; the $G \times G$-orbits in $X \times X$ are products of G orbits in X.

End Dec 9 additions.

Need $F: \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g} \times \mathfrak{g}$ s.t.



$$\begin{tikzcd} \mathfrak{g} \times \mathfrak{g} \arrow[rr, "M"] \arrow[d, "(1\times\sigma\times 1)(1\times F)"'] \arrow[drr, "\text{preserves invariant measures?}"] & & \mathfrak{g} \\ \mathfrak{g}\times\mathfrak{g}\times\mathfrak{g}\times\mathfrak{g} \arrow[rr, "C \times C"'] & & \mathfrak{g} \times \mathfrak{g} \arrow[u, "m"'] \end{tikzcd}$$

where $\sigma(x,y) = (y,x)$ and

$$C(x,y) = e^{-y} x e^{y}$$
$$= e^{-\mathrm{ad}\, y} x$$

Why half-densities?

A measure theory lemma: Given a commutative triangle

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

, and measures μ on X and ν on Y s.t. $\nu = T_* \mu$,

is $T_* E(\mu | p) = E(\nu | q)$? (Seems obvious & easy)