Drinfel'd's Lemma

October 15, 08
7:50 PM

**Lemma.** If $\sum_{i=1}^{m} [Y_i, P_i] = 0$, where the $P_i$ are Lie polynomials in $Y_1, \ldots, Y_m$, then there exists exactly one $f \in \mathcal{F}(Y_1, \ldots, Y_m)$ such that $\partial f / \partial Y_i = P_i$ for all $i$.

**Proof.** The usual connection between polynomials and symmetric multilinear functions allows us to restrict ourselves to the case that $P_i$ does not contain $Y_j$ while $P_i$, $\ldots$, $P_m$ and $f$ are linear in $Y_i$. In this case, if $f$ exists, then $f = (Y_i, P_i)$. Conversely, if $f = (Y_i, P_i)$, then $\partial f / \partial Y_i = P_i$ for all $i$. Indeed, put $Q_i = P_i - \partial f / \partial Y_i$. Then $Q_i = 0$ and $\sum_i [Y_i, Q_i] = 0$. For $i > 1$ write $Q_i$ in the form $R_i(\text{ad} Y_2, \ldots, \text{ad} Y_m) Y_i$, where $R_i$ is an associative polynomial. Then $\sum_{i=1}^{m} u_i R_i(u_2, \ldots, u_m) = 0$, and therefore $R_2 = \cdots = R_m = 0$. 

**Better proof at 2011-02**

$$0 \rightarrow A_n^0 \xleftarrow{\alpha} A_n^w \xrightarrow{s} \text{Lie}_n \rightarrow 0$$

$$0 \rightarrow A_{n+1}^0 \xrightarrow{\alpha} A_{n+1}^w \xrightarrow{s} \text{Lie}_{n+1} \rightarrow 0$$

**Following Drinfel’d, let’s do the degree $(0, 1, 1, \ldots, 1)$ case first:**

**Claim:** if $x_1, \ldots, x_n$
For $\deg T = (1, \ldots, 1)$, then $T = 0$

$T_i \in R^{(1, \ldots, 1)}$

$T_i$'s head not on $x_j$

**Question**

$T_i \in \text{Lie}(x_1, \ldots, x_n, y)$

$\deg_{x_i} T_j = 1 - \delta_{ij}$

$\deg_y T_i = 1$

$\sum [x_i, T_i] = 0 \implies \forall i \ T_i = 0.$

**Claim**

The map $\text{ASS}(x_1, \ldots, x_n) \rightarrow \text{Lie}(x_1, \ldots, x_n, y)$

via

$x_{i_1} \cdots x_{i_k} \rightarrow \text{ad}_{x_{i_1}}(\text{ad}_{x_{i_2}}(\cdots \text{ad}_{x_{i_k}}(y)))$

is injective.

**Proof**

Further compose with the inclusion of $\text{Lie}(x_1, x_n)$

into $\text{ASS}(x_1, \ldots, x_n, y)$. The result is the map

$\alpha = (Tx_i) \rightarrow dy s(\alpha)$.

Now $\alpha$ may be read to the left of the $y$ in the term in which there's nothing to the right of the $y$. 