Drinfel'd's Lemma

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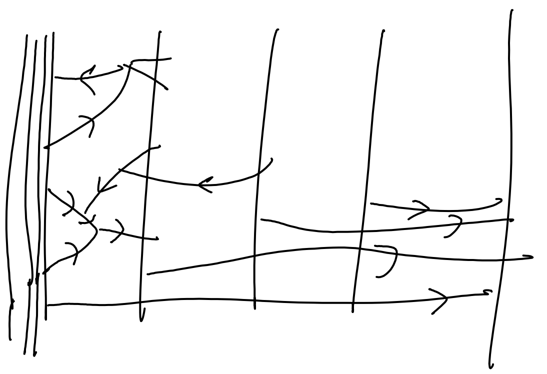
Lemma. *If* $\sum_{i=1}^{m}[Y_i, P_i] = 0$, *where the* P_i *are Lie polynomials in* $Y_1, \dots, Y_m$, *then there exists exactly one* $f \in \mathcal{F}(Y_1, \dots, Y_m)$ *such that* $\partial f/\partial Y_i = P_i$ *for all* i.

Proof. The usual connection between polynomials and symmetric multilinear functions allows us to restrict ourselves to the case that P_1 does not contain Y_1, while $P_2, \dots, P_m$ and f are linear in Y_1. In this case, if f exists, then $f = (Y_1, P_1)$. Conversely, if $f = (Y_1, P_1)$, then $\partial f/\partial Y_i = P_i$ for all i. Indeed, put $Q_i = P_i - \partial f/\partial Y_i$. Then $Q_1 = 0$ and $\sum_i [Y_i, Q_i] = 0$. For $i > 1$ write Q_i in the form $R_i(\mathrm{ad}\, Y_2, \dots, \mathrm{ad}\, Y_m)Y_1$, where R_i is an associative polynomial. Then $\sum_{i=2}^{m} u_i R_i(u_2, \dots, u_m) = 0$, and therefore $R_2 = \dots = R_m = 0$. ●



Better proof at 2011-02

$$0 \to \mathcal{A}_n^p \xrightarrow{\alpha} \vec{\mathcal{A}}_n^{wp} \xrightarrow{s} \mathcal{L}ie_n \to 0$$

$\uparrow$ exact?

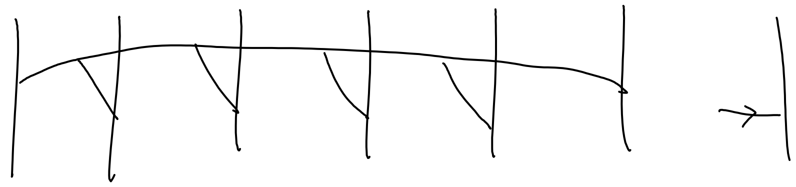
$D \mapsto D(\sum x_i)$

$$\begin{array}{ccccccccc} 0 \to & \mathcal{A}_n^p & \xrightarrow{\alpha} & \vec{\mathcal{A}}_n^{wp} & \xrightarrow{s} & \mathcal{L}ie_n & \to 0 \\ & \downarrow \Delta_1 & & \downarrow \Delta_1 & & \downarrow \Delta_1 & \\ 0 \to & \mathcal{A}_{n+1}^p & \xrightarrow{\alpha} & \vec{\mathcal{A}}_{n+1}^{wp} & \xrightarrow{s} & \mathcal{L}ie_{n+1} & \to 0 \end{array}$$

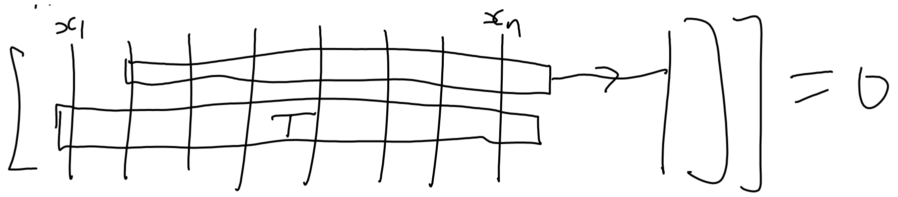
merge colours q

No commutativities

Following Drinfel'd, let's do the degree $(1,1,1,\dots,1)$ case first:



Claim if



for $\deg T = (1, \dots, 1)$, then $T = 0$

$T \in \vec{A}^{wp}_{(1,\dots,1)}$

T's head not on x_1

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question $T_i \in Lie(x_1, \dots x_n, y)$ $\quad \deg_{x_i} T_j = 1 - \delta_{ij}$, $\deg_y T_i = 1$

$$\sum [x_i, T_i] = 0 \Rightarrow \forall i \; T_i = 0 \; ?$$

claim The map $ASS(x_1 \dots x_n) \longmapsto Lie(x_1 \dots x_n, y)$

via

$$x_{i_1} \dots x_{i_k} \longmapsto ad_{x_{i_1}}(ad_{x_{i_2}}(\dots ad_{x_{i_k}}(y))\dots)$$

is injective.

proof Further compose with the inclusion of $Lie(x_1 \dots x_n, y)$ into $ASS(x_1 \dots x_n, y)$. The result is the map

$a = (\prod x_{i_k}) \longmapsto a' y S(a'')$. Now a can be read to the left of the y in the term in which there's nothing to the right of the y.