The problem: Given \( f \in k[x] \),
\[ a_n T(x - \beta_i) \]
where the roots \( \beta_i \) are distinct

Goal: Determine \( \{ \beta_i \} \) without computing the roots.

The classical discriminant
\[ D(f) = \prod_{1 \leq i < j \leq n} (\beta_i - \beta_j)^2 \]
(a poly in \( a_0 \ldots a_n \))

Says geometry
\[ d \ell = \text{Proj } K[a_0 \ldots a_n] = \mathbb{P}^n \]

Identify pts in \( d \ell \) with degree \( n \) poly, up to scalar multiples

Let \( f = a_0 x^n + a_1 x^{n-1} y + \ldots + a_n y^n \in O(1,n) \)
\[ = \prod_{i=1}^{n} (x; x - \beta_i, y)_{k_i} \]

Consider \( \ell \subseteq d \ell \times \mathbb{P}^1 = \text{Proj } K[x_0 : y_0] \)

"Incidence Variety"
\[ \{ (f, [x_0 : y_0]) : [x_0 : y_0] \text{ is a root of } f \} \]
\[ = Z(f) \quad \dim \ell_1 = \dim d \ell = n \]

Likewise,
\[ \ell^k \geq i_k = 0 \ldots \text{ as } \exists \nu \in \mathbb{Q} \text{ is a root } \]

\[ \dim i_k = n - k + 1 \]

The \(i_k\)'s are manifold: \(i_k = \mathbb{Z} \left( \text{all } (k-1)\text{'st derivative} \right) \left( \text{of } F \text{ w.r.t. } x, y \right) \left( \text{Sections of } O(1)^{n-1} \right) \)

(no need look at lower derivatives because of
\[ NF = x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y}, \text{ so if RHS vanishes} \]

Consider \( p_k: \mathcal{H}^{x \langle k \rangle} \rightarrow \mathcal{H}, \Sigma_k = p_k(i_k) \)

\[ \Sigma_1 = \Sigma \text{ as every polynomial has a root.} \]
\[ \mathcal{H} = \Sigma_1 \supset \Sigma_2 \supset \ldots \supset \Sigma_n \text{ all inclusions of codimension 1.} \]

E.g. \( n = 4, k = 2 \)

\[ \text{two double roots.} \]

So \( p_k \) is a desingularization of \( \Sigma_k \).

"The normalization".

Idea: Write down presentation matrices for \( O(i_k) \) as a module over \( O(\Sigma_k) \)

\[ \text{These matrices we want} \]

\[ \mathcal{H} \rightarrow \mathcal{O}(\Sigma_k) \rightarrow \mathcal{O}(i_k) \rightarrow 0 \]

\[ p_k = 0 \]

\[ \text{note: } \]

\[ p_k: i_k \rightarrow \Sigma_k \text{ is a finite morphism and is generically } 1-1 \text{ (hence birational).} \]

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continue to a full resolution; example:

\[ 0 \to \mathcal{O}(-2, -2v) \to \mathcal{O}(-1, 1-n) \to \mathcal{O}^2 \to \mathcal{O} \to \mathcal{O} \to 0 \]