Hovinen on The root Structure of Polynomials

September-05-08
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The problem Given $f \in k[x]$, (char $k = 0$)

$f = a_0 x^n + \ldots + a_n \quad a_0 \neq 0$ or $f = a_0 \prod (x - \alpha_i)^{k_i} = a_0 \prod (x - \beta_i)$ (roots w/ multiplicity)

α_i's are (distinct) roots in a splitting field of f

Goal Determine $\{k_i\}$ without computing the roots.

$$D(f) = \prod_{i<j} (\beta_i - \beta_j)^2 \quad \left(\begin{array}{l}\text{can be written as} \\ \text{a poly in } a_0 \ldots a_n\end{array}\right)$$

The classical discriminant

$\Updownarrow$

tells us whether $\exists i$ s.t. $k_i > 1$.

Some geometry

$$\mathcal{H} = \text{Proj } k[a_0 \ldots a_n] = \mathbb{P}^n$$

identify pts in $\mathcal{H}$ with degree n polys, up to scalar multiples

Let $F = a_0 x^n + a_1 x^{n-1} y + \ldots + a_n y^n \in \mathcal{O}(1,n)$

$$= \prod_i (\alpha_i x - \beta_i y)^{k_i} \leftarrow \text{so multiplicities still make sense.}$$

(α_i, β_i different than before)

Consider

$$I_1 \subseteq \mathcal{H} \times \mathbb{P}^1 \quad (\mathbb{P}^1 = \text{Proj } k[x, y])$$

$$= \{(F, [x_0 : y_0]) : [x_0, y_0] \text{ is a root of } F\}$$

$$= Z(F)$$

The "Incidence Variety"

$\dim I_1 = \dim \mathcal{H} = n$

Likewise,

$\mathcal{H}\times\mathbb{P}^1 \supseteq I_K = \left\{ \dots : \begin{array}{l}[x_0, y_0] \text{ is a root} \\ \text{of } F \text{ of mult} \geq k\end{array} \right\}$

$\dim I_K = n-k+1$

The I_K's are manifold: $I_K = Z\left(\begin{array}{l}\text{all } (k-1)\text{'st derivative} \\ \underbrace{\text{of } F \text{ w.r.t. } x \,\&\, y}_{\text{sections of } \mathcal{O}(1,\, n-k-1)}\end{array}\right)$

(no need look at lower derivatives because of

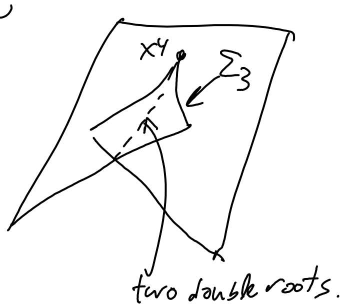
$$nF = x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y}, \quad \text{so if RHS vanishes, so does LHS})$$

Consider $p_{\mathcal{H}} : \mathcal{H}\times\mathbb{P}^1 \to \mathcal{H}$, $\Sigma_k := p_{\mathcal{H}}(I_K)$

$\Sigma_1 = \mathcal{H}$ as every polynomial has a root.

$\mathcal{H} = \Sigma_1 \supset \Sigma_2 \supset \dots \supset \Sigma_n$ all inclusions of codimension 1.

E.g. $n=4$, $k=2$



note:
$p_K : I_K \to \Sigma_K$
is a finite morphism
and is generically
1–1
(hence birational)

So p_K is a de-singularization of Σ_K.

"the normalization".

Idea: Write down presentation matrices for

$\mathcal{O}_{I_K}$ as a module over $\mathcal{O}_{\Sigma_K}$

These matrices we want!

$$\begin{tikzcd} \mathcal{O}_{\Sigma}^{\oplus n_1} \arrow[rr, "(\ )"] \arrow[dr] & & \mathcal{O}_{\Sigma_K}^{\oplus n_0} \arrow[r] & \mathcal{O}_{I_K} \arrow[r] & 0 \\ & K \arrow[ur] \arrow[dr] & & & \\ 0 \arrow[ur] & & 0 & & \end{tikzcd}$$

continue to a full resolution; example:

$$0 \to \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2,2-2n) \to \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,1-n) \longrightarrow \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}^{\oplus 2} \longrightarrow \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} \longrightarrow \mathcal{O}_{I_2} \longrightarrow 0$$