Another Attempt

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Question Find a function f of x and y so that in $U\left(L/[[L,L],[L,L]]\right)$ $(L:=FL(x,y))$,

$$e^x e^y = \exp(x+y+H(f)) \qquad \left(\begin{array}{l}\text{where } H(f):=H_{[x,y]}(f) \\ :=\mathrm{ad}(f)[x,y]\end{array}\right.$$

This time, instead of looking at the flow, we will study scattering. $\left(J(x)=\frac{1-e^{-x}}{x}\right)$

LHS: $e^{-y}e^{-x}xe^{x}e^{y}=e^{-y}xe^{y}=e^{-\mathrm{ad}\,y}x$

$$=x+\frac{e^{-\mathrm{ad}\,y}-1}{\mathrm{ad}\,y}[y,x]=x+J(\mathrm{ad}\,y)[x,y]$$

$$=x+H(J(y))$$

while

$$e^{-y}e^{-x}y\,e^{x}e^{y}=e^{-y}\left(y+H(-J(x))\right)e^{y}$$

$$=y+H\left(e^{y}J(x)\right)$$

RHS:

$$e^{-x-y-H(f)}x\,e^{x+y+H(f)}=e^{-\mathrm{ad}(x+y+H(f))}x$$

$$=x+H\left(\frac{1-e^{-x-y}}{x+y}(y+xf)\right)$$

Aside: For any g — Verify!

$$g(\mathrm{ad}(x+y+H(f)))(x)=$$
$$=g(0)\cdot x+H\left(\frac{g(x+y)-g(0)}{x+y}(-1-xf)\right)$$

comparing the x-scatterings we get:

$$\frac{1-e^{-y}}{y}=\frac{1-e^{-x-y}}{x+y}(1+xf)$$

So $F = \frac{x+y}{x(1-e^{-x-y})} \cdot \left(\frac{1-e^{-y}}{y} + \frac{e^{-x-y}-1}{x+y} \right)$

$$= \frac{(x+y)(e^{x+y}-e^{x})}{xy(e^{x+y}-1)} - \frac{1}{x}$$

almost?

Kurlin says:

Theorem 1.1. *Under* $L \to \widehat{\bar{L}}$ *the Hausdorff series* $H = \ln(e^X e^Y)$ *maps onto* $\bar{H} = X + Y + \frac{1}{y}\left(1 - \frac{e^x - 1}{x} \cdot \frac{x+y}{e^{x+y}-1}\right)[XY]$, *where the operator acting on* $[XY]$ *is considered as a commutative series in* x, y.

What would it take to find an A-T F?

Scattering by e^{x+y}:

$$e^{-x-y} x e^{x+y} = e^{-\mathrm{ad}(x+y)} x = \ldots \quad \left(\begin{matrix}\text{more or less}\\ H(J(x+y))\end{matrix}\right)$$

Scattering by $\exp H(g)$:

$$e^{-H(g)} x e^{H(g)} = e^{-\mathrm{ad}\, H(g)} x = \ldots \quad \left(\begin{matrix}\text{more or less}\\ x + H(xg)\end{matrix}\right)$$